Convex Methods for Blind Equalisation in QAM Transmission Systems

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Statement of Originality

The information presented in this thesis is the result of original research, and has not been submitted for the purposes of obtaining a higher degree at any other university or institution.

Much of the work in this thesis has been published or has been submitted for publication in refereed journals:

1. K. Yamazaki and R.A. Kennedy. Reformulation of linearly-constrained adaptation and its application to blind equalization. *IEEE Trans. on Signal Processing*, 1994. to appear.

2. K. Yamazaki and R.A. Kennedy. Blind equalization for QAM systems based on general linearly-constrained convex cost functions. *Intl. Journal of Automatic Control and Signal Processing*, 1993. under review.

A number of papers have been published in conference proceedings. Some of the material covered in these papers overlaps with that covered in the publications above:

1. K. Yamazaki, R.A. Kennedy, and Z. Ding. Globally convergent blind equalization algorithms for complex data systems. In *IEEE International Conference on Acoustics, Speech, and Signal Processing ICASSP-92*, volume 4, pages 553–556, San Francisco, U.S.A., March 1992.

2. K. Yamazaki, R.A. Kennedy, and Z. Ding. Candidate admissible blind equalization algorithms for QAM communication systems. In *IEEE International Conference on Communications ICC-92*, pages 1518–1522, Chicago, U.S.A, June 1992.

3. R.A. Kennedy, K. Yamazaki, Z. Ding, S. Verdú, and S. Vembu. Approaches to blind equalization: Open problems. In *4th IFAC Intl. Symp. on Adaptive Systems in Control and Signal Processing ACASP-92*, pages 627–632, Grenoble, France, July 1992.

4. K. Yamazaki and R.A. Kennedy. On globally convergent blind equalization for QAM systems. In *International Conference on Acoustics, Speech & Signal Processing ICASSP-94*, Adelaide, Australia, April 1994. to appear.

The research described in this thesis is the result of a collaborative effort with my PhD supervisor Dr Rodney A. Kennedy and Dr Zhi Ding. A small portion of work on the effect of data correlation on the convergence of our blind equalisation algorithm was the result of collaboration between Dr Rodney A. Kennedy and Dr Wolfgang Sauer-Greff. However, the majority of the work, approximately 70%, is my own.

Ken Yamazaki

Ken Yamazaki
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[1] But, seriously, I've always wanted the letters "DR" to precede my name on my *MasterCard*

[2], sticky rudders, termites in the hull boards, malicious mermaids, ...

[3] Romans 8:28-30.

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Abstract

Foremost in this work is the proposal and analysis of a *blind equalisation algorithm* suitable for *quadrature amplitude modulation (QAM)* transmission over mixed-phase channels. The algorithm adapts the equaliser parameters by performing a *stochastic gradient descent* on a *convex cost function* while maintaining a *linear constraint* on the equaliser parameters. The algorithm is the result of successive generalisations of an earlier one proposed for real PAM transmission by Kennedy, Vembu, and Verdu.

Using methods from *convex analysis*, a novel *geometric view* of the cost minimisation problem is formulated that permits analysis of the *true* convergence behaviour of the algorithm under *both* infinite and finite equaliser parametrisations. In contrast, this feature is not shared by other blind algorithms such as the *Sato*, *Godard*, *Constant Modulus Algorithm (CMA)*, and *Benveniste-Goursat-Ruget (BGR)* algorithms.

Within this geometric framework, the convex cost, QAM symbol constellation, linear parameter constraint and channel define a *convex polytope* and *hyperplane* in the combined channel-equaliser or "*total*" parameter space. The polytope is a closed and bounded set with a boundary or *cost level surface* containing all of the total parameter settings that give a fixed value of the cost. The value of the cost determines the size of the polytope. The *vertices* of the polytope correspond to equaliser parameter settings that produce zero *intersymbol interference (ISI)* at the output of the equaliser. The hyperplane corresponds to a linear constraint in the total parameter space and its orientation is determined by the channel and parameter constraint. The linearly-constrained cost minimisation problem is then alternatively posed in the total parameter space as one of increasing the size of the polytope, *i.e.*, value of the cost, until the polytope touches the hyperplane, *i.e.*, the constraint is satisfied. Generally, the polytope touches the hyperplane on a vertex which implies global minimisation of the constrained cost

by a zero-ISI equaliser parameter setting. With this interpretation, we derive and prove succinctly the result that, in all but a few degenerate channel-constraint combinations, the algorithm will always converge to a zero-ISI equaliser parameter setting irrespective of the specific initialisation of the parameters.

The phenomenon of *nonunique minimisation* originally observed in the algorithm of Kennedy *et al.* is intuitively understood and generalised to the case of QAM transmission. A geometric view of this phenomenon establishes it as the case where a degenerate orientation of the hyperplane, *i.e.*, degenerate channel-equaliser combination, induces a situation where the polytope touches the hyperplane on an *edge* or *face* of the polytope.

We also develop a *canonical representation* of the linearly-constrained equaliser which is the cascade of a *prefilter* and an equaliser with a constraint on a single parameter. The ideas that follow from this representation and the geometric framework may facilitate the development of methods to circumvent the nonuniqueness problem such as the use adaptive linear constraints or special nonlinear parameter constraints.

We also provide a broad classification and assessment of the various approaches to blind equalisation in order to identify those approaches that might offer some benefits over the others. The results are used to justify the particular approach that we have taken.

Contents

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Glossary

QAM	quadrature amplitude modulation
PAM	pulse amplitude modulation
ISI	intersymbol interference
MESA	modified error signal algorithm
CMA	constant modulus algorithm
SGD	stochastic gradient descent
BGR	Benveniste-Goursat-Ruget
LTI	linear time-invariant
TLA	three-letter acronym
VLSI	very large scale integration
DSP	digital signal processor
SAR	synthetic aperature radar
LMS	least mean square
BIBO	bounded-input bounded-output
EWGI	equalisation without gain identification

iid	independent identically distributed
$\mathbb{C}$	complex plane
$\mathbb{C}^L$	complex inner product space of dimension L
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\mathcal{A}$	symbol alphabet (or constellation)
$\mathcal{M}$	compact set of the minimisers of $\mathcal{J}(\cdot)$
$\mathcal{T}$	total parameter space or t-space (Banach space)
$\mathcal{H}$	hyperplane in the t-space
$\mathcal{H}^+$	half-space excluding the origin in the t-space
$\mathcal{P}_\alpha$	cost level surface set of scale α
$\partial\mathcal{P}_\alpha$	boundary set of $\mathcal{P}_\alpha$
l_p	l_p-space: $x \triangleq \{\ldots, x_{-1}, x_0, x_1, \ldots\} \in l_p$ if $\sum_{i=-\infty}^{\infty} \lvert x_i\rvert^p < \infty,\ p \geq 1$
$*$	convolution: $y = a * b \Rightarrow y_k = \sum_{i=-\infty}^{\infty} a_i b_{k-i}$.
$\mathcal{Q}(\cdot)$	QAM quantisation function
$\mathrm{Re}\{\cdot\}$	real part
$\mathrm{Im}\{\cdot\}$	imaginary part
$\mathrm{E}[\cdot]$	expectation operator
$\Psi(\cdot)$	cost function
$\psi(\cdot)$	gradient of cost function
$\mathcal{J}(\cdot)$	mean cost function
$\boldsymbol{H}(\cdot)$	Hessian
$\arg(\cdot)$	phase angle
$\mathrm{sgn}(\cdot)$	signum function
$\langle\cdot,\cdot\rangle$	complex inner product: $\langle \boldsymbol{A}, \boldsymbol{B}\rangle \triangleq \mathrm{Re}\left\{\boldsymbol{A}^H \boldsymbol{B}\right\},\ \boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}$
$\mathrm{proj}_{\boldsymbol{C}}(\cdot)$	projection onto $\boldsymbol{C}$: $\mathrm{proj}_{\boldsymbol{C}}(\boldsymbol{X}) \triangleq \boldsymbol{C}\langle \boldsymbol{C}, \boldsymbol{X}\rangle / \langle \boldsymbol{C}, \boldsymbol{C}\rangle$
$\lVert\cdot\rVert_p$	l_p norm: $\lVert x\rVert_p \triangleq \left(\sum_{i=-\infty}^{\infty} \lvert x_i\rvert^p\right)^{(1/p)},\ p \geq 1$
$p_x(\cdot)$	probability density function of the random variable x
$\{\cdot\}$	sequence: $\{x_k\}$ denotes sequence with elements x_k
$\mathcal{D}(\cdot)$	measure of the gain γ of the equaliser output relative to the channel input

Chapter 1

Introduction

In this first chapter, we introduce and motivate the notion of *blind equalisation* and discuss its relevance to the general problem of equalisation in digital transmission systems. It is intended to give the reader a broad familiarisation with the key concepts of blind equalisation without being overly rigorous in its descriptions. Of equal importance is our introduction of a new approach to blind equalisation that uses ideas from convex analysis. This approach will be become the main focus of the thesis and will be dealt with in subsequent chapters.

In Section 1.1 we describe the problem of blind equalisation as a special case of the more general problem of *blind deconvolution*. We list the particular requirements of blind equalisation that set it apart from other forms of blind deconvolution. The conventional *training sequence* method of equalisation is reviewed to contrast it with blind equalisation which characteristically does not require a training sequence. We motivate the blind equalisation problem by listing some applications and discussing the overall position of blind equalisation in the general area of equalisation. Two different classes of blind equalisation algorithms are mentioned of which one, namely, the class of *modified error signal algorithms*, is argued as being more practical. Section 1.2 previews our new approach to blind equalisation using convex methods and foreshadows the benefits and major results that have been developed in the remainder of the thesis. In Section 1.3 we give a brief outline of the thesis.

1.1 An Introduction to Blind Equalisation

The general term *blind deconvolution* refers to the process of recovering or forming an *estimate* of an *unknown* signal applied to an *unknown linear time-invariant (LTI)* system by observing and operating on the system output *only*. Such a situation is depicted in Figure 1-1. There we assume that the operations are in discrete-time with time index k. The *impulse response* of the unknown LTI system is denoted by the

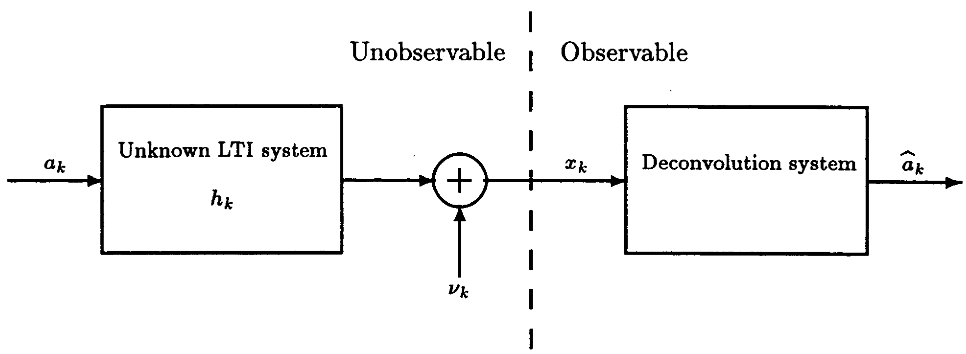


Figure 1-1: Blind Deconvolution.

discrete-time sequence $\{h_k\}$. The unknown input signal sequence is denoted by $\{a_k\}$. The output of the system is corrupted by an additive noise sequence, $\{\nu_k\}$. The result, $\{x_k\}$ is the output sequence observed by the deconvolution system and is given by

$$x_k = a_k * h_k + \nu_k \tag{1.1.1}$$

where $*$ denotes discrete-time convolution. The objective of the deconvolution system is to employ some *adaptive algorithm* to produce an estimate of the unknown signal, $\{a_k\}$, based solely on the observable quantity $\{x_k\}$. The *blind* aspect stems from the constraint that the deconvolution algorithm does not have *explicit* knowledge of the input signal and hence cannot form any measure of the *actual* error between the input signal and the estimate as can be done by other more conventional nonblind algorithms. For example, the simplest measure of the actual error is

$$e_k = \widehat{a}_k - a_k. \tag{1.1.2}$$

Blind algorithms may rely instead on some foreknowledge of a suitable *stationary* statistical property of the input signal and attempt to restore that property when forming an estimate of the input signal.

Applications of blind deconvolution are abundant. For example, there is homomorphic filtering in image processing[1], deconvolution in the analysis of seismic data [2, 3], estimation of Doppler parameters for spaceborne synthetic aperture radar (SAR) [4] and debluring of images from radiotelescope arrays [5]. In contrast to the actual problem we consider in this thesis, all of these applications do not require the restored signal to be available in real-time and hence the deconvolution operations may be done *off-line* on blocks of stored received data.

In the context of digital transmission between a transmitter and receiver, *blind* or *"self-recovering" equalisation* is yet another application or special case of blind deconvolution. Here the unknown system input is an information bearing *symbol sequence* sent from the transmitter and the unknown system itself is a communication channel that induces *intersymbol interference (ISI)* at its output. An *equaliser* in the receiver serves the role of the deconvolution system. There are several general characteristics that distinguish the blind equalisation problem from other forms of blind deconvolution and are also important for the discussions in this thesis. The following descriptions will be brief as the details will be covered in subsequent sections and in Chapter 2.

Linear Deconvolution System

The *equaliser* is a *linear* discrete-time deconvolution system. It typically takes the form of a *linear transversal filter* because of its simple design and ease of implementation.

Linear Mixed-Phase Channel

The unknown linear channel is modelled as a *causal linear discrete-time filter* and may possess a *mixed-phase* property. The ramification of the mixed-phase property is that the *stable transversal filter* model of the *inverse* of such a filter will be *noncausal.*

Fixed Finite Alphabet

The input symbol sequence $\{a_k\}$ consists of symbols from a *fixed finite alphabet*, *i.e.*, *finite set*, or *"constellation"*. The final recovery of the individual symbols of the sequence will involve a *quantisation* stage following the equaliser. It will quantise each sample of the equaliser output to one of the symbols in the constellation. This implies that the equalisation operation need only restore the input symbols to a degree that is sufficient for correct quantisations to be made. In this way, perfect restoration of the symbol sequence is possible.

Algorithm Computational Complexity

The restored signal must be made available in real-time (with little delay) and thus the equalisation must be performed *on-line* on the channel output x_k. Accordingly, blind equalisation algorithms must be computationally simple and lend themselves towards *very large scale integration (VLSI)* or *digital signal processor (DSP)*-based implementations.

Rapid Convergence

The linear channel characteristics may be slowly time-varying. Therefore, a blind equalisation algorithm must converge relatively rapidly in order to track the time-varying parameters of the channel.

The last two requirements impose a limit in practice on the computational complexity of blind equalisation algorithms and thus needs serious consideration in the development of the algorithm. Such is typically not the case with other forms of deconvolution where *off-line* batch processing on large mainframe computers suffices.

To further motivate our discussion of the blind equalisation problem, we need to briefly present the case of *conventional equalisation* in a digital transmission system. As we are only interested in conveying the general concepts, our descriptions, including that of the ISI model, will be simplified.

1.1.1 Conventional Equalisation

Consider the simplified equivalent baseband representation of the equalisation problem depicted in Figure 1-2.

A transmitter sends symbols denoted by the sequence $\{a_k\}$ through a channel that is assumed stable, causal, complex, linear and possibly mixed phase. Its impulse response $\{h_k\}$, $h_k \in \mathbb{C}$, is of length greater than one, that is, $k \in (1, \infty]$. The symbols $\{a_k\}$ are elements of a finite D-element set or *constellation*

$$\mathcal{A} = \{a^{(0)}, a^{(1)}, \ldots, a^{(D-1)}\}, \tag{1.1.3}$$

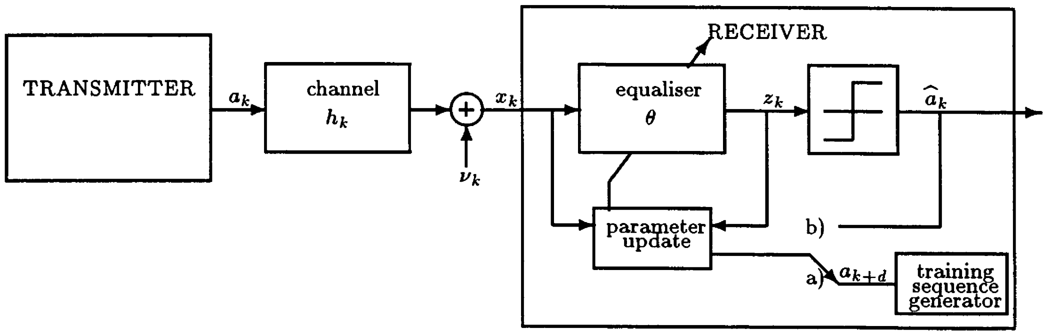


Figure 1-2: Simplified Equivalent Baseband Equalisation Model. a) Training sequence phase. b) Decision-directed mode.

where $a^{(i)} \in \mathbb{C}$, $i \in \{0, 1, \ldots, D-1\}$. The latter name is derived from the pattern observed when all symbols of $\mathcal{A}$ are plotted as points in the complex plane. Perhaps the most common class of complex constellations is that adopted by *quadrature amplitude modulation (QAM)* transmission systems where the real and imaginary parts of the symbol effectively amplitude modulate carrier frequencies that are in a *quadrature* relationship to each other, i.e., a phase difference of $\pi/2$. Two examples of common QAM constellations are shown in Figure 1-3.

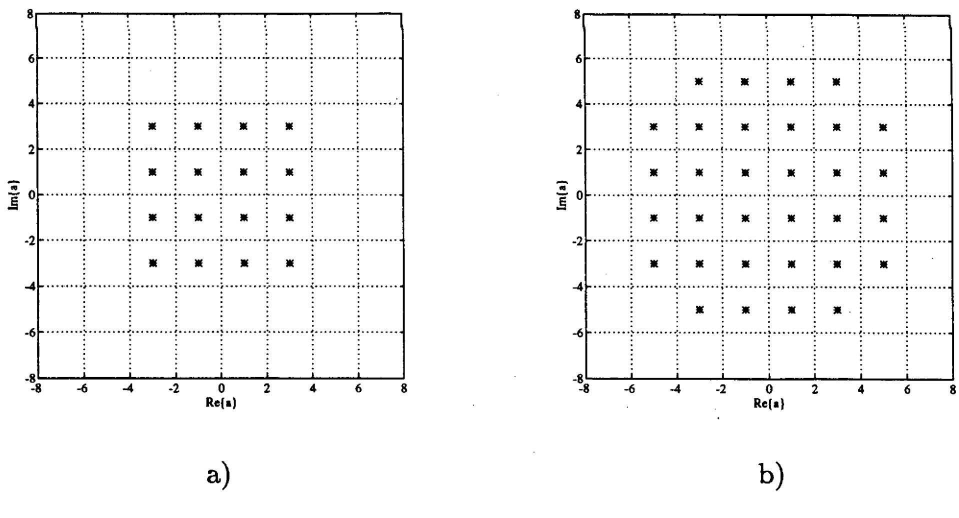


Figure 1-3: Typical QAM constellations. a) 16-QAM b) 32-QAM.

The channel impulse response $\{h_k\}$ is generally not known at the start of transmission because the characteristics of a specific channel typically fluctuate over time primarily due to varying environmental conditions. Alternatively, the channel itself may represent one of many possible routes to the receiver that is selected at the time a link is established between a transmitter and receiver. The output of the linear channel at time k, $x_k \in \mathbb{C}$, can be represented as a convolution between $\{a_k\}$ and $\{h_k\}$ and hence consists of a linear combination of the input symbol a_k and scaled versions of adjacent symbols, *i.e.*, $\{a_{k-1}, a_{k-2}, \ldots\}$. The component due to the adjacent symbols is the undesirable *intersymbol interference* or *ISI.* Mathematically,

$$x_k = \sum_{i=0}^{\infty} h_i a_{k-i} = h_0 a_k + \underbrace{\sum_{i=1}^{\infty} h_i a_{k-i}}_{\text{ISI}} . \tag{1.1.4}$$

It must be noted that the description of the ISI in (1.1.4) is simplified because it assumes that no delay can be tolerated between the input symbol sequence $\{a_k\}$ and the symbol sequence that will be recovered from the channel output sequence $\{x_k\}$. For example, if a delay of one is permitted, then the term $h_1 a_{k-1}$ can be considered the desired quantity and the remaining terms of the sum including $h_0 a_k$ will constitute the ISI.

The role of the receiver is to recover the channel input sequence $\{a_k\}$ from the received sequence $\{x_k\}$ while possibly incurring some tolerable delay Δ_t in the process. A naive approach would be to *directly quantise* each x_k using a *nearest neighbour quantiser* or *decision device* $\mathcal{Q}(\cdot)$ such as that indicated by the last block in the chain in Figure 1-2. The output of the quantiser $\widehat{a}_k$ can be expressed, in terms of its input, in this case, x_k, by

$$\widehat{a}_k = \mathcal{Q}(x_k) = \arg\min_{a \in \mathcal{A}} |x_k - a| \tag{1.1.5}$$

where $|\cdot|$ denotes the complex modulus function. Now, consider the simple case where $h_0 = 1$ and the symbol sequence $\{a_k\}$ is independent identically distributed (iid). A sufficient condition for the decision device to output correct decisions is

$$|\text{Peak ISI}(k)| = \max_{\{a_k\}} \left| \sum_{i=1}^{\infty} h_i a_{k-i} \right| < \frac{d_{\min}}{2} \tag{1.1.6}$$

where $d_{\min}$ is the *minimum distance* between any two symbols in the constellation defined by

$$d_{\min} \triangleq \min_{i \neq j,\ i,j \in \{0,1,\ldots,p-1\}} |a^{(i)} - a^{(j)}|. \tag{1.1.7}$$

The condition on the ISI (1.1.6) for correct decisions to be made by the quantiser is termed the *open-eye* condition. This comes from the fact that, for a binary antipodal signaling system, the corresponding *eye diagram* of the input to the quantiser resembles an open eye despite there being no such resemblance for QAM signaling. For our purposes, an *eye diagram* is a plot of successive values of an output on the complex plane. Visually, the eye diagram indicates an "*open-eye*" condition when it appears as a hazy version of the original constellation with a clustering of points about each original constellation point and a distinct separation between each cluster group.

1.1.1.1 The Equalisation Objective

Clearly, there may exist channels for which (1.1.6) is not satisfied, *i.e.*, a "*closed-eye*" condition exists. In these cases, the direct quantisation of the channel output $\{x_k\}$ results in incorrect decisions. To deal with the closed-eye situation, the receiver contains an *equaliser* (as shown in Figure 1-2): typically, an *adaptive linear transversal filter* placed before the decision device. The equaliser performs a convolution between the channel output $\{x_k\}$ and its parameters $\{\theta_i\}$, $i = 0, \ldots, L-1$, namely,

$$z_k = \sum_{i=0}^{L-1} \theta_i x_{k-i} \tag{1.1.8}$$

where L is the number of transversal equaliser parameters.

The objective of the equaliser is to adapt its parameters $\{\theta_i\}$ so that the ISI in the equaliser output $\{z_k\}$ is reduced sufficiently so as to achieve an open-eye condition. In the absence of noise, *i.e.*, $\nu_k = 0 \; \forall \; k$, the decision device implementing the quantisation function $\mathcal{Q}(\cdot)$ can then produce correct decisions in the following sense:

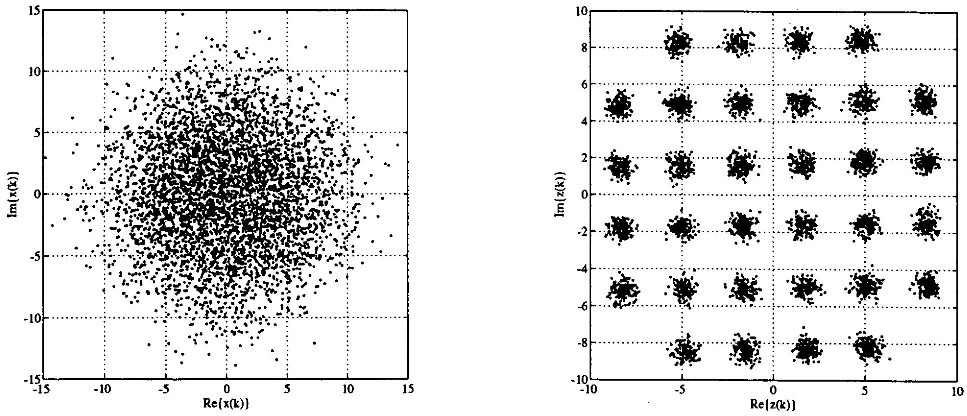
$$\begin{aligned} \widehat{a}_k = \mathcal{Q}(z_k) &= \arg\min_{a \in \mathcal{A}} |z_k - a| \\ &= a_{k-\Delta_t}. \end{aligned} \tag{1.1.9}$$

Figure 1-4 gives a typical eye-diagram of the equaliser output before and after the eye has been opened assuming the symbols belong to a 32-QAM constellation.

From a system identification viewpoint the linear equalisation process may be regarded as adapting the equaliser parameters $\{\theta_i\}$ so that they form the approximate *inverse of the channel* $\{\widetilde{h}_k\}$. The inverse of the channel has the property that it nullifies the effect of the channel when it is placed in cascade with the channel, that is,

$$h_k * \widetilde{h}_k = \delta_k \tag{1.1.10}$$

where $\{\delta_k\}$ is the Kronecker delta sequence. In the ideal case where no noise is present and the equaliser parameters are able to *perfectly* represent the inverse of the channel,



a) b)

Figure 1-4: Opening the eye for 32-QAM symbols. a) Before equalisation b) After equalisation.

the equaliser output is given by

$$\begin{aligned} z_k &= a_k * h_k * \theta_k = a_k * h_k * \widetilde{h}_k \\ &= a_k * \delta_k \\ &= a_k. \end{aligned} \tag{1.1.11}$$

This result indicates a *perfect* recovery of the channel input symbols $\{a_k\}$. However, the *practical* situation is one where noise *is* present in the channel output $\{x_k\}$ and the equaliser has an insufficient number of parameters to *perfectly* model the inverse of the channel. Thus, at best, the equaliser can only provide an *approximation* of the channel inverse that is sufficient to open the eye diagram.

As indicated by Figure 1-2, the adaptation of the equaliser parameters is governed by a *parameter update algorithm* which may utilise the channel output $\{x_k\}$, its own output $\{z_k\}$, and any *a priori* information about the channel and transmitted symbol sequence. From this point onwards our discussion will illuminate the distinction between *conventional* and *blind* equalisation.

1.1.1.2 Conventional Equalisation Algorithms

Conventional adaptation of the equaliser parameters typically occurs during an initial start-up or *training phase* of transmission between the transmitter and the receiver and during subsequent phases where readaptation is required. During such a phase, normal data transmission is halted for the transmission of a pseudorandom symbol sequence or *training sequence.* The exact sequence is known *a priori* by both the transmitter and receiver. At the same time the training sequence is transmitted through the channel an identical sequence is generated locally in the receiver to provide the parameter update algorithm with the knowledge of the actual transmitted sequence. This is illustrated in Figure 1-2 with the switch between the training sequence generator and the parameter update block at position a). This mutual knowledge of the transmitted symbol sequence is exploited to form the *prediction error*: the error between the estimate of the transmitted symbol at the output of the equaliser z_k and the actual symbol transmitted provided locally by the training sequence generator in the receiver. Due to the fact that the transmitted symbol experiences an unavoidable propagation delay Δ_p as it travels through the channel, the estimate z_k will be an estimate of the delayed version of the transmitted symbol, that is, $a_{k-\Delta_p}$. The locally generated sequence must therefore be delayed by the same amount in order for the prediction error to have any meaning. The computation of the prediction error is therefore

$$e_k = z_k - a_{k-\Delta_p}. \tag{1.1.12}$$

The required knowledge of the propagation delay Δ_p in the computation of (1.1.12) implies the need for accurate *synchronisation* between the transmitter and the local generation of the training sequence at the receiver.

To achieve an open-eye condition, conventional algorithms attempt to minimise some form of the prediction error. For example, given a stochastic process model for the symbol sequence $\{a_k\}$, the well-known *least mean square (LMS) algorithm* minimises $\mathrm{E}\left[e_k^2\right]$ (where $\mathrm{E}\left[\cdot\right]$ denotes *expectation* over all symbol sequences $\{a_k\}$). The expected squared error $\mathrm{E}\left[e_k^2\right]$ is a quadratic function of the equaliser parameters that is uniquely minimised by an open-eye equaliser parameter setting provided there is a sufficient number of parameters. The LMS algorithm adopts the simple *stochastic gradient descent (SGD)* technique to accomplish the minimisation. It performs a gradient descent on the *instantaneous* squared error e_k^2 which can be interpreted as a noisy estimate of its expected form, nevertheless, the estimate is adequate if the gradient descent step-size parameter μ is sufficiently small. The resulting equaliser parameter update recursion for the ith parameter is

$$\theta_i(k+1) = \theta_i(k) - \mu e_k x_{k-i} \tag{1.1.13}$$

where e_k is given by (1.1.12). Adaptation continues until the open-eye condition arises at which point the adaptation is halted and normal data transmission is allowed to proceed.

In order to track the slowly time-varying characteristics of the channel and maintain the open-eye condition during the transmission of normal data, the equaliser is switched to a *decision-directed mode* whereby the parameter update algorithm uses the decision of the quantiser, which is now correct, to form the prediction error. This corresponds to the switch in Figure 1-2 set to position b).

To end our discussion, we emphasise that central to the workings of any training sequence-based conventional algorithm is its minimisation of some form of the *prediction error* which is a *true* measure of the error between the estimated and actual transmitted symbols.

1.1.2 Blind Equalisation

As we have seen in Section 1.1.1, conventional equaliser adaptation depends upon having a particular phase of transmission during which normal transmission of data symbols is halted for the transmission of a special training sequence which is synchronised to the local generation of the same sequence in the receiver. However, in many situations, it is costly, highly impractical, or in fact impossible to have such a phase. *Blind equalisers* circumvent this limitation as they characteristically do not require a training sequence. Instead, they operate during the normal transmission of data the values of which are not known *a priori*. This situation is highlighted by the equivalent baseband blind equalisation model in Figure 1-5.

Note that the training sequence generator of the conventional equaliser is absent. The qualification of *"blind"* in this context refers to the fact that the parameter update algorithm does not have explicit knowledge of the transmitted symbol sequence and therefore *cannot* form the actual prediction error e_k given by (1.1.12). Figure 1-5, indicates that, like conventional equalisation, the blind equaliser can be operated in two distinct modes. Initially, before adaptation has occurred, the closed-eye condition

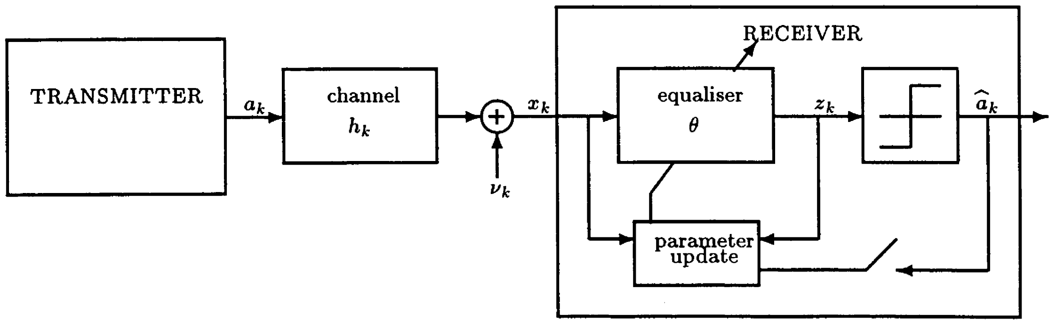


Figure 1-5: Equivalent Baseband Blind Equalisation Model

generally exists at the output of the equaliser and hence the output of the decision device is not correct. While this condition exists, the switch shown is left open and the parameters are updated by the blind parameter update algorithm which has at its disposal the channel output x_k and the equaliser output z_k only. After the parameters are updated to obtain the open-eye condition, the equaliser may then enter a *decision-directed mode* as in the case of conventional equalisation to track the slowly time-varying characteristics of the channel. This is accomplished by closing the switch and allowing the parameter update algorithm to access the now correct output of the decision device to form the prediction error. Thus, after the blind update algorithm achieves the open-eye condition, the equaliser is operated in a manner indistinguishable from a conventional one.

1.1.2.1 Applications of Blind Equalisation

Blind equalisation algorithms have the distinguishing feature that they are suitable in situations where it is impractical or impossible to halt normal data transmission for the transmission of a training sequence. Examples of those situations that arise in practice are now given.

A blind equalisation algorithm was first proposed in [6] for multilevel digital transmission networks. In the situation described, after a transmission route to a receiver has been established, the route may be suddenly switched to an alternate one if it experiences excessive signal degradation. As this occurs after the connection has been established, the equaliser must readapt to the resulting new channel conditions without a retransmission of the training sequence from the transmitter.

High-speed multipoint digital modem networks are characterised by a single control modem and multiple tributary modems on a shared line. Tributary modems are allowed to transmit only when polled by the control modem. During initial network initialisation, the control modem broadcasts a training sequence to the tributary modems so that their equalisers can be adapted for their respective channels. In the cases where a channel to a tributary modem changes significantly or a tributary modem was not powered on during the network initialisation phase, the control modem must halt normal data transmission to retrain that tributary modem thereby reducing the overall data throughput of the network. Clearly, in such cases, it is beneficial for the equalisers to have the ability to retrain themselves without the need for a training sequence. A blind equalisation algorithm motivated by this need is described in [7]. More recent systems using 64 and 128-QAM signaling constellations are described in [8].

During severe multipath fading in line-of-site (LOS) digital microwave radio systems, equalisers must reconverge while receiving an unknown data sequence transmitted through an unknown channel. The common zero-forcing (ZF) equalisation algorithm reconverges slowly and thus makes attractive the use of faster blind algorithms [9].

Finally, blind equalisation may also be used in covert signal intercept operations where, quite understandably, the "enemy's" training sequence is not known *a priori* and synchronisation is impossible.

It is evident from the above that the need for blind equalisation has arisen out of specific transmission situations. However, it can be argued that blind equalisation represents a *philosophical ideal* form of equalisation. The main reason why equalisation is necessary is because the transmission channel is usually not known *a priori*. Since this unknown channel is effectively determined by the location of the receiver with respect to the transmitter, it makes sense to place the burden of equalisation entirely on the shoulders of the receiver. This notion is supported, for example, by the digital modem network broadcast situation described earlier and the case of digital mobile radio transmission where the receiver's location with respect to the transmitter may be changing constantly. Viewed at the transmission function level, little if any part of the transmitter should be relegated to ensure that the receiver is equalising its signal. This includes the transmission of a training sequence and establishing the frequently difficult synchronisation between the transmitter and receiver. Since a blind equaliser is able to train itself during normal data transmission and does not require any additional resources of the transmitter, it can be implemented entirely as a receiver function. In this light, blind equalisation may be viewed as representing the opposite extreme to training-sequence based equalisation. Therefore, it can be seen that blind equalisation represents at least a step closer to the philosophical ideal.

It must be noted that *current* blind equalisation algorithms tend to converge more slowly than their non-blind counterparts and, therefore, the latter form is still chosen in applications requiring fast startup or recovery from rapidly changing channel characteristics. For example, to combat the time-varying characteristics of the channel in a wireless application, periodic transmissions of a training sequence integrated with normal data transmission may be required [10]. However, there is no apparent reason why blind algorithms should *characteristically* converge at a lower rate. Although blind algorithms operate with lower quality information about the transmitted symbol sequence, it is the appropriate exploitation of this information that may ultimately reduce this discrepancy in convergence rates. In addition, the resulting high administrative overhead of implementing frequent training periods incurs the penalty of lowering the overall rate of useful information transmitted.

Hybrid Equalisation Scheme

The penalty incurred by periodic training sequence-based schemes provides impetus for a *hybrid* adaptation scheme that is motivated by the following observation. Periodic training sequence-based adaptation relies on the transmission of a training sequence that is of a sufficient length. That is, it must be sufficient to give the adaptation algorithm time to reestablish an open-eye condition from an almost closed-eye condition that may exist at the start of the training period. Adaptation is limited to those periods when a training sequence is transmitted. Blind adaptation, on the other hand, can take place at *any* time and, in particular, during those periods when normal data is transmitted and the training sequence-based adaptation is consequently turned off. A possible hybrid scheme may then entail *two-phase* equaliser parameter adaptation which is an augmentation of the current method of periodically transmitting a training sequence. Training sequence-based adaptation occurs during the transmission of the training sequence and blind adaptation occurs during the transmission of normal data. Therefore the adaptation phase is continually alternating. The purpose of the blind phase is to achieve or maintain a level of ISI that is at least lower than that which would exist without the blind adaptation. In this way, less time is required by the subsequent training sequence-based phase to reestablish the open-eye condition. This fact allows the length of the training sequence phase, and hence the training sequence itself, to be reduced. This has the desirable effect of increasing the useful information rate. Therefore, applied to possible hybrid approaches, blind equalisation techniques might have an impact in the conventional equalisation arena by offering improvements in information throughput for systems which till now have relied solely on relatively long training sequences.

1.1.3 Blind Equalisation Algorithms

As in the general case of blind deconvolution, a blind equalisation algorithm may take the approach of seeking to restore a stationary property of the transmitted symbols that corresponds to the restoration of the symbols themselves. Candidate properties may be the invariant modulus of the symbols of *phase shift keying* (PSK) constellations or perhaps the squareness property of a square *quadrature amplitude modulation* (QAM) constellation. This *property restoration* aspect can be expressed formally as the minimisation of some *mean cost function* $\mathcal{J}(\theta)$ which indicates the degree to which the invariant property is restored at the output of the equaliser. Note that by using the *mean* of the cost, the algorithm is seeking to restore the invariant property at the output of the equaliser in an *averaged* sense, that is, averaged over all possible transmitted symbol sequences. Consequently, the mean cost function is a deterministic function of the equaliser parameters $\{\theta_i\}$. A central issue in the design of the mean cost function is the desirable property that its stable global minima correspond to zero-ISI equaliser parameter settings.

As in the case of the LMS algorithm for conventional equalisation, the minimisation of the mean cost function can be done on-line using the stochastic gradient descent (SGD) technique. As an example of an algorithm adopting the above approach, we give the parameter update recursion of the popular blind algorithm known as the *Constant Modulus Algorithm (CMA)* [11, 12] which performs a stochastic gradient descent to minimise a special cost function that reflects the restoration of the *modulus* of symbols from a constellation with a constant modulus such as 4-QAM and 8-PSK. If we assume that the modulus of the symbols is 1 then the mean cost function is given by

$$\mathcal{J}(\theta) = \frac{1}{4}\mathrm{E}\left[(|z_k|^2 - 1)^2\right]. \tag{1.1.14}$$

The form of (1.1.14) illustrates the property restoration idea in that the cost is minimised when θ is chosen such that the resulting modulus of the equaliser output matches the modulus of the original transmitted symbols, *i.e.*, 1.

The SGD technique performs a gradient descent on the *instantaneous* form of the mean cost, that is,

$$\Psi(z_k) = \frac{1}{4}(|z_k|^2 - 1)^2. \tag{1.1.15}$$

The resulting equaliser parameter update recursion for the ith equaliser parameter is given by

$$\theta_i(k+1) = \theta_i(k) - \mu[(|z_k|^2 - 1)z_k]x_{k-i}. \tag{1.1.16}$$

If we compare the blind CMA update recursion of (1.1.16) to that of the nonblind LMS algorithm in (1.1.13), we discover that the only difference is, for the blind update, the prediction error $e(k)$, given by (1.1.12), has been replaced by a blind error quantity $e'(k) = (|z_k|^2 - 1)z_k$. Adopting the terminology of [13] we refer to this alternate error quantity as a *modified error signal* to distinguish it from the prediction error $e(k)$ of conventional equalisation which requires explicit knowledge of the transmitted symbol sequence. Likewise, we refer to all blind algorithms of this type as *modified error signal algorithms* or *MESA*'s.

MESA's represent an important class of blind algorithms that have received much attention in recent years owing to their relative low computational complexity and subsequent amenability to hardware realisations. However, before we continue further on the topic of MESA's, we briefly digress to mention another class of blind deconvolution algorithms, called *polyspectral algorithms*, that have been proposed for blind equalisation.

1.1.3.1 Polyspectral Algorithms

Polyspectral or *Higher Order Statistics (HOS)* techniques have already been proposed for blind identification of nonminimum-phase systems [14, 15, 16] with motivation derived from applications such as seismic signal processing [2, 3]. However, these methods are *off-line* techniques and require relatively large amounts of data to be collected for batch processing.

More recently, adaptive algorithms based on these techniques have been proposed for blind equalisation [17, 18]. These approaches exploit a known relationship between the fourth-order *cumulants* (or their complex cepstrum called the *tricepstrum*) of the output of a linear possibly mixed-phase FIR channel and the zeros of the channel. The main advantage of dealing with fourth-order cumulants is that they generally preserve the *phase* information of the channel output which is otherwise lost by using second order statistics alone. Typical assumptions are that the input can be modelled as a white, independent identically distributed (i.i.d.) non-Gaussian process and that additive noise at the channel input has a zero-mean Gaussian distribution. The channel is also permitted to be slowly time-varying.

The approach begins with the computation of estimates of the tricepstral coefficients of the channel output. With this information, the known relationship between the zeros of the channel and the tricepstrum of the channel output can be expressed by a linear overdetermined system of equations that can be solved iteratively to effectively give the amplitude and phase spectra of the channel. The constant channel gain factor is recovered separately by a simple *automatic gain control* algorithm which typically attempts to match the power of the channel output with the power of the transmitted symbols.

Generally, these schemes perform very well, however, they have serious drawbacks that currently limit their practicality. Specifically, estimation of the fourth-order cumulants of the channel output is required and is accomplished by time-averaging over a window of the output sequence. Due to the slowly time-varying nature of the channel and hence the nonstationarity of the channel output, the window length is effectively limited by the requirement that the output over this window be relatively stationary for accurate time-averaged estimates. This imposes an overall limit to the accuracy of these schemes as the variance of the estimates increase with shorter window lengths. Furthermore, these methods naturally require larger storage and significantly higher computational complexity. This makes them presently unsuitable for practical implementations on current hardware architectures which ultimately must perform the adaptation on-line at high-speed and with low delay.

Due to their current unsuitability for practical implementations we will not concern ourselves with the polyspectral approach for the remainder of this thesis.

1.1.3.2 Modified Error Signal Algorithms

We have seen in Section 1.1.3 that *modified error signal algorithms (MESA's)* derive their name from the fact that they use a *modified error signal* which does not depend on explicit knowledge of the transmitted symbols and is therefore unlike the prediction error used by conventional equalisation algorithms. MESA's adopt a *mean cost function* which is the expected value (over all transmitted symbol sequences) of a special function of the equaliser output. The mean cost is designed such that its stable minima ideally correspond to zero-ISI equaliser parametrisations. We have also seen that the simple *stochastic gradient descent* method is used to adapt the equaliser parameters *on-line* to one of these minima thereby attempting to achieve the equalisation objective. Details of the MESA approach are given in Chapter 3, Section 3.1.

The attractiveness of the MESA approach stems from the fact that, computationally, the algorithm is of the same complexity as the popular *Least Mean Square* (LMS) algorithm which has been implemented in hardware for conventional equalisation. MESA algorithms differ mainly by the particular *cost function* used which is usually some measure of the restoration of some stationary property of the transmitted symbol constellation. For example, many of the well known blind equalisation algorithms, namely, the *Sato* [6],*Godard* [7], *Constant Modulus Algorithm (CMA)* [11], and the *Benveniste-Goursat-Ruget (BGR)* [19] algorithms can be classified as MESA's. Ideally, the global and local minima of the mean cost function should correspond to an equaliser parametrisation that gives a zero-ISI condition. However, it has been shown that all of those algorithms mentioned adopt costs which, under practical finite equaliser parametrisations, possess stable local minima that do not correspond to open-eye equaliser parameter settings [20, 21, 13]. In those works, ill-convergence to local minima corresponding to closed-eye parameter settings is demonstrated. In these cases, an equaliser parameter initialisation policy must be enforced in order to avoid adaptation to such local minima.

The term *admissibility* was introduced to be a main objective of blind equalisation algorithms [22, 23]. In brief, the admissibility of an algorithm is its ability to guarantee convergence to the vicinity of an ideal parameter setting from arbitrary parameter initialisations. *Ideal parameter settings* in our context refers to equaliser parameter settings that give the zero-ISI condition. With this definition, the well-known algorithms mentioned clearly fail to be admissible.

Despite their current lack of *admissibility*, MESA's thus far represent the only category of algorithms that are computationally simple enough for the high-speed hardware implementations required by telecommunication applications. This fact warrants further research into developing *admissible* MESA's.

1.2 A New Approach to Blind Equalisation using Convex Methods

In this section, we preview our new approach to blind equalisation which is the focus of this thesis. Although the approach is detailed in Chapter 5, it is motivated and founded in the preliminary results established in Chapters 3 and 4.

Our new approach encompasses both the methods of *design* and *analysis* that culminate in the development of a blind equalisation algorithm for QAM systems that is *admissible* under both *infinite* and *finite* equaliser parametrisations. Furthermore, our approach produces an algorithm whose convergence properties can be intuitively understood under finite equaliser parametrisation. The admissibility and *amenability to analysis* under both *infinite* and *finite equaliser parametrisations* are features that are lacking among many of the popular blind equalisation algorithms due in part to a *local minima problem* that arises when the equaliser is finitely parametrised.

Using a MESA formulation, our approach entails adopting a cost function that is a *convex* function of the equaliser output and also the equaliser parameters. Unique to our approach is the imposition of a *linear* constraint on the equaliser parameters. The use of a convex cost function permits the exploitation of existing methods and results from *convex analysis* to study the convergence properties of our subsequent algorithm. Convex cost functions offer many potential advantages over nonconvex cost functions. To list a few:

1. They are unimodal by nature and possess a global minimum. Local characterisation of the minimum is equivalent to a global characterisation. This makes

the analysis of the convergence of the associated algorithm relatively easy as the convergence is not affected qualitatively by the particular state of initialisation of the equaliser parameters.

2. They do not lose their convexity property under linear constraints on the equaliser parameters. Finite parametrisation of the equaliser is equivalent to a series of simple linear constraints on the equaliser parameters and, therefore, it does not affect the convexity property of the cost function.

3. Suppose that the minimum of the convex cost identifies a zero-ISI equaliser parameter setting under infinite equaliser parametrisation. Under finite equaliser parametrisation, it can be shown that the minimum of the cost can identify a zero-ISI equaliser parameter setting to an arbitrary degree of accuracy provided the number of parameters is made sufficiently large [24]. An algorithm based on a convex cost is therefore *amenable to analysis* in the sense that its ideal properties can be deduced relatively easily assuming an infinite equaliser parametrisation. Then these ideal properties can be achieved to an arbitrary degree of approximation under a finite parametrisation as the number of parameters is made large. This feature is not generally shared by other nonconvex costs which, under finite equaliser parametrisation, may possess local stable minima that do not correspond to open-eye equaliser parameter settings. This creates the possibility that the finitely parametrised equaliser may converge to a closed-eye equaliser parameter setting.

In our approach we develop a particular convex cost that is suitable for QAM systems and also formulate a novel *geometric view* of the cost minimisation problem which provides an intuitive understanding of the ideal and nonideal convergence behaviour of our subsequent algorithm. This geometric view assumes an infinite equaliser parametrisation to allow the study of the ideal convergence properties of our algorithm. However, given our previous discussion, this view is also useful in the practical case of a finitely parametrised equaliser because we are guaranteed arbitrarily close behaviour to the ideal with a sufficient number of equaliser parameters.

To facilitate our later analysis it is useful to introduce the concept of the combined channel-equaliser parameter space or *total parameter space.* In this space, the convex cost, QAM constellation, linear parameter constraint, and the channel define two geometric objects in an abstract manner, namely, a *cost polytope* and a *constraint hyperplane.* The boundary of the polytope is a *level surface* of the convex cost containing all of the total parameter settings that give a fixed value of the cost. The value of the cost determines the size of the polytope. We will show that the *vertices* of the polytope correspond to equaliser parameter settings that produce zero ISI at the output of the equaliser. The constraint hyperplane defines a linear constraint in the total parameter space and its orientation is determined by the channel and linear equaliser parameter constraint. The linearly-constrained cost minimisation problem is then alternatively posed in the total parameter space as one of increasing the size of the polytope, *i.e.*, value of the cost, until the polytope touches the hyperplane, *i.e.*, the constraint is satisfied. Generally, the polytope touches the hyperplane on a vertex which implies global minimisation of the constrained cost by a zero-ISI equaliser parameter setting. With this interpretation, we derive and prove succinctly the result that, in all but a few degenerate channel-constraint combinations, the algorithm will always converge to a zero-ISI equaliser parameter setting irrespective of the specific initialisation of the parameters. This is a global convergence result that establishes the *admissibility* of our algorithm.

Remarks

1. Although the establishment of the geometric view is based on abstract methods from convex analysis, the proof of the admissibility of our algorithm (which exploits this view) is short and intuitive. Convexity of our cost ensures that this result is applicable to a finite equaliser parametrisation that has a sufficient number of parameters. Therefore, we stress the important result that we have developed an algorithm that is *amenable to analysis* even *under a practical finite equaliser parametrisation.*

2. Our geometric view reveals a potential *nonunique minimisation phenomenon* and establishes it as the case where a degenerate orientation of the hyperplane, *i.e.*, degenerate channel-equaliser combination, induces a situation where the polytope touches the hyperplane on an *edge* or *face* of the polytope. With this interpretation the phenomenon is deemed nongeneric in that the probability of its

occurrence is low.

3. Our geometric view provides an intuitive understanding of the factors that affect the convergence of our algorithm. This facilitates the study of the effect that other modifications may have on the algorithm such as the use of multiple linear constraints or adaptive linear constraints.

1.3 Outline of Thesis

In this chapter, we have introduced the problem of blind equalisation in an informal manner and have motivated it in the general context of equalisation in digital transmission systems. In our discussion of blind equalisation algorithms we have found that the class labelled *modified error signal algorithms (MESA's)* is the most practical given their relative computational simplicity and ease of implementation in hardware. We also remarked that most of the popular blind equalisation algorithms are in this class. However, none of them possess the desirable criterion of *admissibility* under practical finite equaliser parametrisations. We concluded this chapter with a preview of a new approach to blind equalisation based on convex methods which among other things offers a blind equalisation algorithm that is admissible under both infinite and finite equaliser parametrisations. Furthermore, the algorithm is amenable to analysis under both parametrisations. This analysis is significantly aided by the formulation of a novel geometric view of the cost minimisation problem. The remainder of this thesis is dedicated to developing this new approach.

In Chapter 2, we formulate the blind equalisation problem in detail. We give our assumptions of the input symbol constellation, channel, and equaliser models. We also discuss the factors that must be considered in the practical implementation of blind equalisers such as the issue of finite equaliser parametrisation. The justification for our new approach begins in Chapter 3 where we collate the major contributions concerned with the design and analysis of MESA's from the literature and organise them into a proper classification that allows for the identification of less desirable approaches. The work described in Chapter 4 is direct consequence of the results of the survey in Chapter 3. The preliminary work on the design and analysis of our algorithm is presented as a series of generalisations of an earlier algorithm that adopts a convex cost function suitable for real PAM systems. In addition, a cost design philosophy and a precursor to the geometric view of our new approach is presented. In Chapter 5 our blind equalisation algorithm suitable for QAM systems is presented as the final generalisation of the earlier algorithm for PAM systems. There, we finally refine our new approach by adopting the geometric view of the cost minimisation problem. Arising out of our geometric view is a proof of the admissibility of our algorithm. We also present a practical implementation of our algorithm and discuss other practical issues such as the effect of channel coding and noise on the convergence behaviour of our algorithm. We support our discussions with simulations of our algorithm. Finally, in Chapter 6, we summarise the main ideas established in this thesis and provide a chapter by chapter account of the major results obtained. We finish our discussion by proposing a few topics for future research that may complement the work in this thesis.

Chapter 2

Blind Equalisation Problem

In this chapter, we formulate the blind equalisation problem in detail. Specifically, in Sections 2.1 to 2.3, we give our assumptions of the input constellation, channel, and equaliser models. We then present the factors that must be considered in the practical implementation of any blind equalisation algorithm in Section 2.4. The *blind equalisation problem* is then formally stated in Section 2.5.

2.1 Input Constellation Model

Consider the simplified baseband channel and blind equaliser configuration depicted in Figure 2-1.

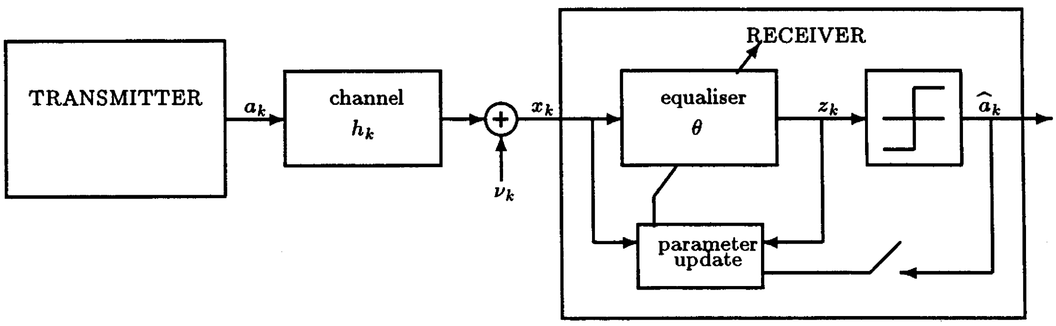


Figure 2-1: Equivalent Baseband Blind Equalisation Model

The transmitted sequence $\{a_k\}$ consists of symbols from a complex QAM symbol constellation set, $\mathcal{A}$. We make the following assumptions about $\{a_k\}$.

Assumption 2.1 Probabilistic Version *All finite subsequences of the input sequence $\{a_k\}$ have nonzero probability.*

Note that Assumption 2.1 is less constraining than, for example, an *independent identically distributed (iid)* assumption on $\{a_k\}$. For brevity, we will simply say that all sequences have nonzero probability.

A deterministic modelling assumption can alternatively be assumed on the input to complement the above assumption in a nonrandom modeling environment. Our results are valid also for this case.

Assumption 2.2 Deterministic Version *All transitions between data symbols are possible at every time instant.*

For this thesis we shall prefer the *probabilistic version* of the assumption since a *stochastic model* for the input sequence is common in telecommunications. However, both Assumptions 2.1 and 2.2 are sufficient but not strictly necessary for our results.

Assumption 2.3 Stationarity *The symbol sequence $\{a_k\}$ is stationary.*

Assumption 2.3 is sufficient for our analysis as a blind equalisation algorithm tends to operate on the basis that there is some stationary property of the symbol sequence that it can attempt to restore at the equaliser output.

Assumption 2.4 Non-Gaussian Distribution *The symbol sequence cannot have a Gaussian distribution.*

A Gaussian distribution is completely characterised by its second order statistics. If the channel is linear and its input has a Gaussian distribution then the output will also have a Gaussian distribution and, therefore, will not convey any information about the *phase* that the channel response imparts on the input. In this sense, we say that methods based on second order statistics are *phase blind.* The following is an extremely weak (strict) condition on the geometry of the constellation $\mathcal{A}$ and is required later in our development of a *geometric view* of the cost minimisation problem in Chapter 5.

Assumption 2.5 QAM Phase Richness *The QAM symbol constellation is sufficiently rich in phase such that*

$$\max_{a \in \mathcal{A}} Re\{at\} \geq \epsilon|t|, \quad \epsilon > 0 \ \forall \ t \in \mathbb{C} \tag{2.1.1}$$

where t is an arbitrary complex number.

All practical QAM constellations satisfy this condition. For example, it is trivially satisfied when the constellation is comprised of three or more symbols such that the phase angle between adjacent (in phase) symbols is less than π. However, this condition would fail in the case of binary transmission (over a complex channel), but, this dubiously relevant case is easily treated by extension of the methods for in [24]. To illustrate the phase richness condition, Figure 2-2 shows two examples of constellations lacking in phase richness. Note that, in both examples, there exists a phase angle between adjacent symbols that either equals or exceeds π. On the other hand, Figure 2-3 shows an example of a constellation comprised of the minimum of three symbols that satisfies the phase richness condition.

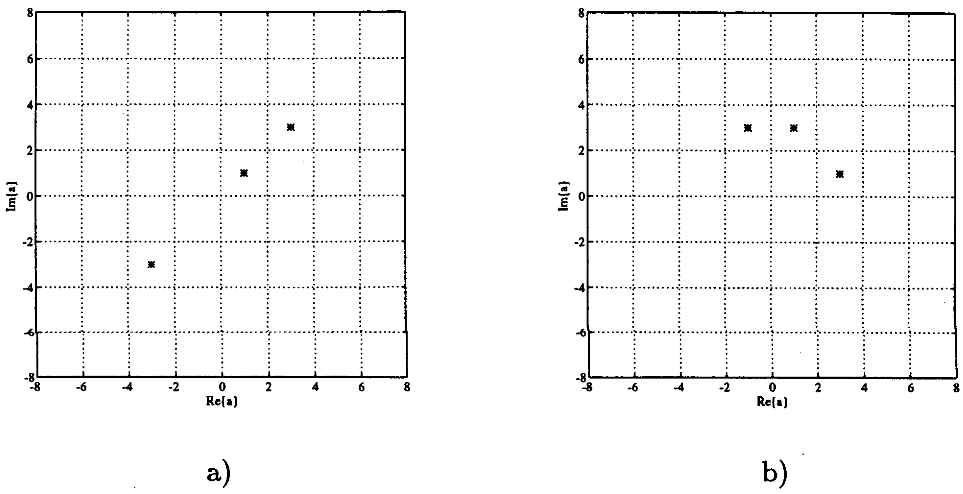


Figure 2-2: Constellations lacking in phase richness. a) Example 1 b) Example 2.

2.2 Channel Model

The complex linear channel is assumed *stable*, possibly *mixed-phase*, and has an impulse response represented by the infinite-length, *absolutely summable* (l_1) sequence $\{h_i\}$ where $h_i \in \mathbb{C}$. The following defines the *mixed-phase* property.

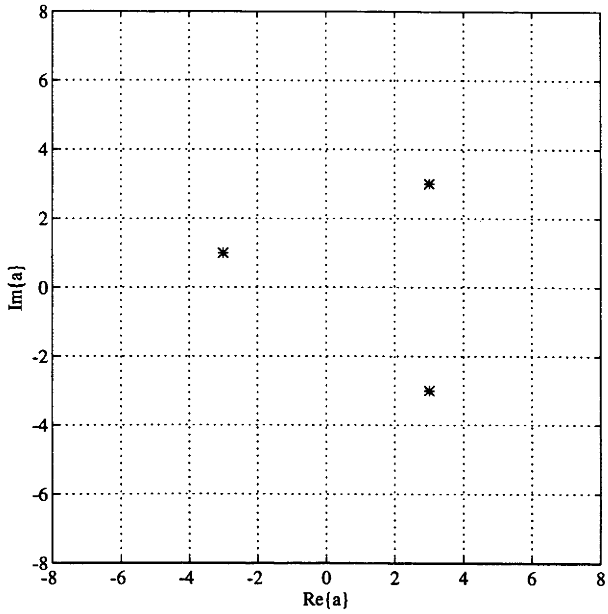


Figure 2-3: A constellation that is sufficiently phase rich.

Definition 2.1 (Mixed-phase) *A linear discrete-time filter with coefficients* $\{h_k\}$, $k \in \mathbb{Z}$ *has a mixed-phase property if its Z-transform,*

$$H(z^{-1}) = \sum_{i=-\infty}^{\infty} h_i z^{-i},$$

has zeros both inside and outside of the unit circle.

A *mixed-phase* channel is unlike a *minimum-phase* (or *maximum-phase*) channel in that the phase response of the former *cannot* be recovered by methods which only exploit the foreknowledge of the second-order statistics of the input signal. These methods are insufficient because second-order statistics do not convey any *phase* or *sign* information. They may only offer recovery of the magnitude information of the channel frequency response. Therefore, with the assumption of a mixed-phase channel, a *blind* cost function that only reflects the restoration of the second-order statistics of the symbol sequence at the output of the equaliser is unsuitable.

The *stable inverse* of a mixed-phase channel is also *noncausal* and this has repercussions on the choice of an appropriate equaliser model. The details will be discussed later in Section 2.3.

The channel will be assumed *causal*, *i.e.*, $h_i = 0$ for $i < 0$, although this is not a necessary assumption. We also impose the restriction that the channel does not possess any zeros on the unit circle as such a channel would have a *channel inverse* impulse response $\{\widetilde{h}_k\}$ that does not decay to zero (as $|k| \to \infty$). The channel inverse response $\{\widetilde{h}_k\}$ has the property that

$$\sum_{i=-\infty}^{\infty} \widetilde{h}_i h_{k-i} = \delta_k \quad \forall\, k \tag{2.2.1}$$

where δ_k is the *Kronecker delta sequence.*

In the absence of noise, the channel output $\{x_k\}$, corrupted with *intersymbol interference (ISI)*, may be expressed as the convolution between the $\{a_k\}$ and $\{h_i\}$ as follows:

$$x_k = \sum_{i=0}^{\infty} h_i a_{k-i}, \quad h_i \in \mathbb{C},\ a_k \in \mathcal{A}. \tag{2.2.2}$$

We defer our treatment of the additive noise signal $\{\nu_k\}$ of Figure 2-1 until Chapter 5 where we remark on its effect on the convergence of our blind equalisation algorithm.

2.3 Equaliser Model

We assume that the equaliser is a *linear transversal filter* with complex parameters $\{\theta_i\}$, $\theta_i \in \mathbb{C}$, $i \in \mathbb{Z}$. The *transversal structure* is most often used in practice because it can be implemented easily in both hardware and in *digital signal processing (DSP)* chip software. Furthermore, the known convergence properties of many proposed blind equalisation algorithms for the simple transversal filter structure still remain poorly understood [23].

As mentioned in Chapter 1, Section 1.1.1.1, the *equalisation objective* may be regarded as adapting the equaliser parameters to identify the approximate *inverse* of the channel in some sense. The double-sided infinite-length equaliser parametrisation $\{\theta_i\}$, $i \in \mathbb{Z}$, is adopted to facilitate the analysis and, more importantly, to model the possibly infinite-length inverses of mixed phase channels that must be realised noncausally to be stable in a bounded-input, bounded-output (BIBO) sense. This arises from the fact that the resulting poles of the inverse lying outside the unit circle must correspond to a region of convergence that contains the unit circle and this occurs when those poles are realised noncausally [25]. Figure 2-4 illustrates the impulse response of a mixed-phase channel and a truncation of the infinite-length stable realisation of its inverse which indicates its double-sided (noncausal) nature.

The equaliser performs a convolution between its parameters and the channel output to form the output sequence $\{z_k\}$, where

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i x_{k-i}, \quad \theta_i \in \mathbb{C}. \tag{2.3.1}$$

An alternate interpretation or representation that we will exploit in our developments in Chapters 4 and 5 is illustrated in Figure 2-5.

Mathematically, we can express the equaliser output by

$$z_k = \sum_{i=-\infty}^{\infty} t_i a_{k-i}, \quad t_i \in \mathbb{C} \tag{2.3.2}$$

where $\{t_i\}$ is defined as the *total* impulse response. In other words, $\{t_i\}$ is the impulse response of the cascade of the channel and equaliser satisfying

$$t_k = h_k * \theta_k = \sum_{i=-\infty}^{\infty} \theta_i h_{k-i} \tag{2.3.3}$$

where $a * b$ denotes the convolution of sequences a and b, and $a_k * b_k$ the kth term in that convolution. Note that if we assume that $\{\theta_i\}$ is double-sided and of infinite length then $\{t_i\}$ is also generally of this form. We may also express the equaliser parameters $\{\theta_i\}$ in terms of the total impulse response parameters $\{t_i\}$ as follows:

$$\theta_k = t_k * \widetilde{h}_k = \sum_{i=-\infty}^{\infty} t_i \widetilde{h}_{k-i}. \tag{2.3.4}$$

2.4 Practical Considerations

Although the double-sided infinite-length parametrisation is adopted to facilitate the *theoretical analysis* of an algorithm, an actual transversal equaliser implementation must possess a *finite* number of parameters and hence can only provide a finite-length approximation of the inverse of the channel. A practical parametrisation for an implementable transversal equaliser is given in delay operator (z^{-1}) form as follows:

$$z^{-N} \sum_{i=-N}^{N} \theta_i z^{-i} \tag{2.4.1}$$

where it is assumed that an adequate approximation is achieved when N is large enough. The parametrisation described by (2.4.1) implies a double-sided truncation of the double-sided infinite-length equaliser parameters followed by the application of a delay of z^{-N} to make them causal. Figure 2-6 illustrates the impulse response of the cascade of the channel of Figure 2-4 with a causalised version of the truncated double-sided channel inverse also shown in Figure 2-4. Note that, as a result of the finite-length approximation, the result is not ideal in the sense of (2.2.1).

Remarks

1. Originally, well-known blind equalisation algorithms based on *multimodal* cost functions such as the *Sato* [6], *Godard* [7], *Constant Modulus Algorithm (CMA)* [11], and the *Benveniste-Goursat-Ruget (BGR)* algorithms [19] were shown to be *admissible* using the assumption of an infinitely parametrised equaliser model. However, all of these algorithms have been shown to be *inadmissible* under *finite* equaliser parametrisations due to the introduction of spurious local minima in the cost function that do not correspond to open-eye equaliser parameter settings [13, 20, 21]. It is shown that this *local minima problem* exists for any value of N in (2.4.1) including the case where the channel inverse is of a finite length and can be exactly modelled by the equaliser. Therefore it *cannot* be assumed that an algorithm will necessarily converge to the vicinity of an open-eye equaliser parameter setting with a sufficient number of equaliser parameters.

2. In our research to be described later the problem of *finite parametrisation* leading to inadmissibility is circumvented by focusing our attention on *convex* cost functions which among other things possess a global minimum and remain convex under finite parametrisation. Therefore, the use of convex cost functions avoids the local minima problem.

2.5 Problem Statement

Referring to Figure 2-1 again, we address the following problem in this thesis.

Blind Equalisation Problem *Consider an unknown, complex, linear, causal, stable, and possibly mixed-phase channel with impulse response* $\{h_k\}$. *The channel input sequence* $\{a_k\}$ *is unknown, though, is assumed to be stationary and comprised of symbols from a QAM constellation* $\mathcal{A}$. *The channel output sequence* $\{x_k\}$ *can be expressed as*

$$x_k = \sum_{i=0}^{\infty} h_i a_{k-i}, \quad \forall\ k. \tag{2.5.1}$$

A linear transversal equaliser with parameters $\{\theta_i\}$ *is placed in cascade with the channel. The equaliser output is given by*

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i x_{k-i}, \quad \forall\ k. \tag{2.5.2}$$

Based on observations of the channel output $\{x_k\}$ *and the equaliser output* $\{z_k\}$, *attempt to adjust the equaliser parameters to reduce sufficiently the ISI, i.e., open the eye, so as to recover the channel input* $\{a_k\}$ *in the following sense:*

$$\mathcal{Q}\left(\frac{z_k}{\gamma}\right) = a_{k-\Delta}, \quad \forall\ k \tag{2.5.3}$$

where $\mathcal{Q}(\cdot)$ *is the QAM quantisation function and* $\gamma \in \mathbb{C}$ *and* $\Delta \in \mathbb{Z}$ *represent a possibly unknown fixed gain and fixed delay of the equaliser output relative to the channel input. The condition (2.5.3) implies that, with respect to the quantisation function* $\mathcal{Q}(\cdot)$,

$$z_k \approx \gamma a_{k-\Delta}, \quad \forall\ k. \tag{2.5.4}$$

Remarks

1. In the case where the magnitude of the fixed unknown gain is not unity then it can easily be recovered by a separate *automatic gain control* stage between the equaliser [26, 24] and the quantiser.

2. The phase component of the gain $\arg(\gamma)$ represents a rotation of the QAM constellation at the equaliser output relative to the channel input. Such a rotation is irrelevant if the symbols are differentially encoded before transmission [27].

3. If the symbol constellation possesses *rotational symmetry* then it is desirable for the quantisation function $\mathcal{Q}(\cdot)$ to reflect this in the following sense:

$$\mathcal{Q}(e^{j\phi}w) = e^{j\phi}\mathcal{Q}(w) \tag{2.5.5}$$

where ϕ belongs to the *set of isomorphic rotations of the symbol constellation* $\mathcal{A}$ and w is the output of the equaliser with the magnitude of the fixed gain equal to unity, that is, $|\gamma| = 1$. For example, the 16-QAM constellation possesses isomorphic rotations $k\pi/2,\ k \in \{0, 1, 2, 3\}$.

4. Given that the transmitted symbols have a stationary character, it is impossible for a blind system (that only observes the output of the channel and equaliser) to determine the absolute delay Δ in (2.5.3). For example, it is impossible to distinguish between a scenario where the symbol sequence is delayed by one symbol period prior to entering the channel and the scenario where the channel itself introduces that delay.

5. A *persistence of excitation* assumption of the channel input may be necessary for the adaptation of the equaliser parameters to an open-eye equaliser parameter setting. This may take the form of an assumption on the properties of the symbol sequence $\{a_k\}$ such as Assumption 2.1.

In the *ideal* case the approximate relation of (2.5.4) becomes

$$z_k = \gamma a_{k-\Delta}, \quad \forall\ k. \tag{2.5.6}$$

This is realised when the transversal equaliser parameters $\{\theta_i\}$ form an appropriately scaled and delayed version of the channel impulse response $\{\widetilde{h}_i\}$ namely,

$$\theta_i = \gamma\widetilde{h}_{i-\Delta} \quad \forall\ i. \tag{2.5.7}$$

We can see that the parameter setting given by (2.5.7) produces an equaliser output given by (2.5.6) as follows. First, we substitute (2.5.7) into the equation for the total impulse response in (2.3.3) and use the property of the inverse channel sequence given by (2.2.1) to simplify the result, yielding

$$\begin{aligned} t_k &= \sum_{i=-\infty}^{\infty} \theta_i h_{k-i} = \sum_{i=-\infty}^{\infty} \gamma\widetilde{h}_{i-\Delta} h_{k-i} \\ &= \gamma\delta_{k-\Delta}, \quad \forall\ k. \end{aligned} \tag{2.5.8}$$

Finally, we substitute the result of (2.5.8) into the expression of the equaliser output in terms of the total impulse response given by (2.3.2). The result is

$$\begin{aligned} z_k &= \sum_{i=-\infty}^{\infty} t_i a_{k-i} = \sum_{i=-\infty}^{\infty} \gamma \delta_{i-\Delta} a_{k-i} \\ &= \gamma a_{k-\Delta}, \quad \forall\, k. \end{aligned} \tag{2.5.9}$$

We call the ideal parameter setting of (2.5.7) a *zero-ISI equaliser parameter setting* because the corresponding equaliser output given by (2.5.6) is not comprised of a component due to adjacent symbols. Note also that (2.5.8) gives the total impulse response that corresponds to a zero-ISI setting.

Using the formulation of the equaliser output in (2.3.2) the equalisation objective given by (2.5.3) can be expressed in terms of a *cursor* and ISI term. If we adopt the following interpretation of the equaliser output

$$z_k = \underbrace{t_\Delta a_{k-\Delta}}_{\text{cursor}} + \underbrace{\sum_{i \neq \Delta} t_i a_{k-i}}_{\text{ISI}}, \quad \forall\, k \tag{2.5.10}$$

then, the equalisation objective can be expressed in the following manner.

$$\mathcal{Q}\left(\frac{z_k}{\gamma}\right) = \mathcal{Q}\left(\frac{t_\Delta a_{k-\Delta}}{\gamma} + \frac{\sum_{i \neq \Delta} t_i a_{k-i}}{\gamma}\right) \tag{2.5.11}$$

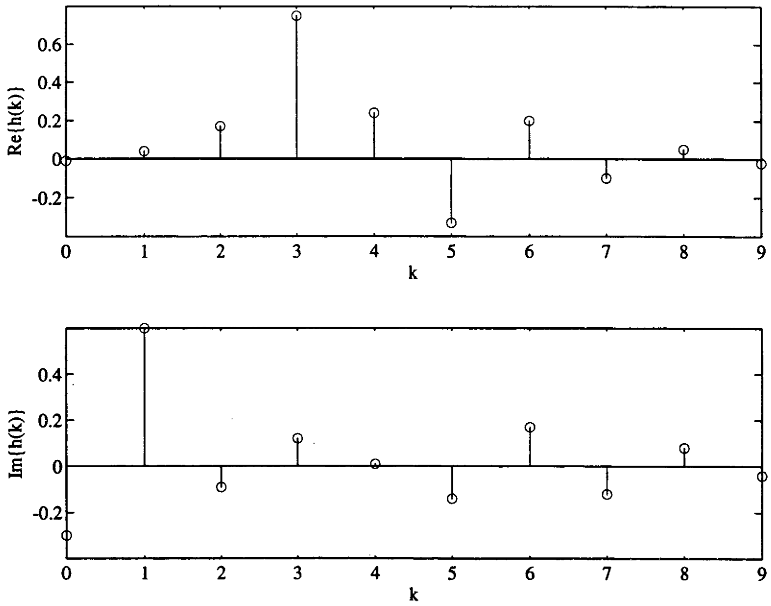
$$= \mathcal{Q}\left(\frac{t_\Delta a_{k-\Delta}}{\gamma}\right) \tag{2.5.12}$$

$$= a_{k-\Delta}, \tag{2.5.13}$$

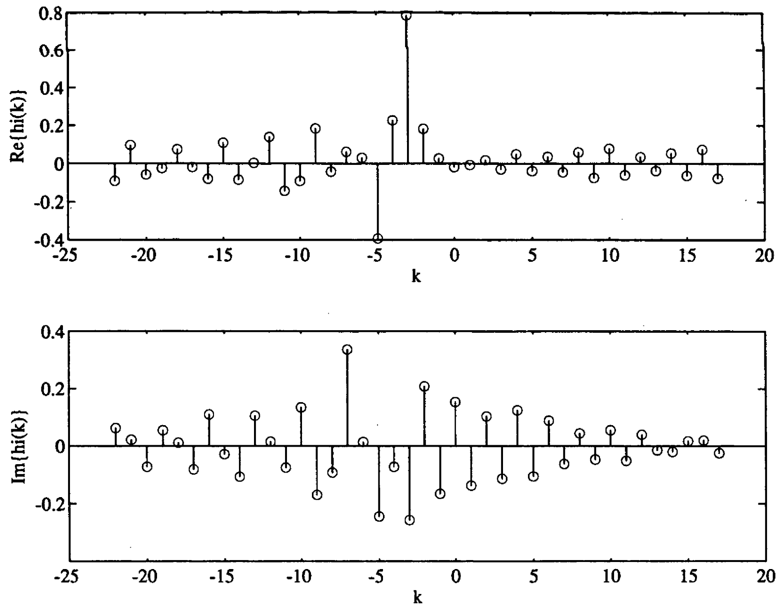
noting that $t_\Delta = \gamma$ by (2.5.8). Equations (2.5.11) to (2.5.13) imply that, with respect to the quantisation function $\mathcal{Q}(\cdot)$, the cursor term "*dominates*" the ISI term. We may also write

$$t_k \approx \gamma \delta_{k-\Delta}, \quad \forall\, k \tag{2.5.14}$$

where $\{\delta_k\}$ is the Kronecker delta sequence. Note that (2.5.14) is just the sufficient version (with respect to the quantisation function $\mathcal{Q}(\cdot)$) of the ideal case given by (2.5.8).



a)



b)

Figure 2-4: a) Impulse response of complex channel. b) Truncated noncausal impulse response of the inverse of the channel.

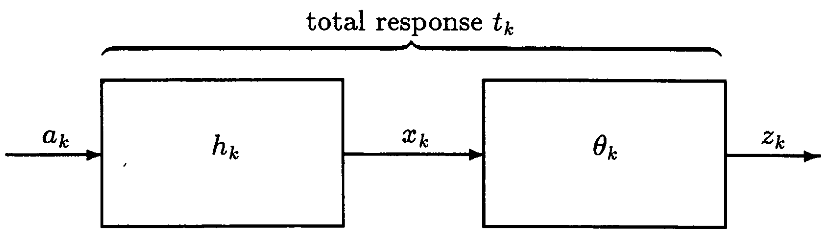


Figure 2-5: Total response of the channel and equaliser.

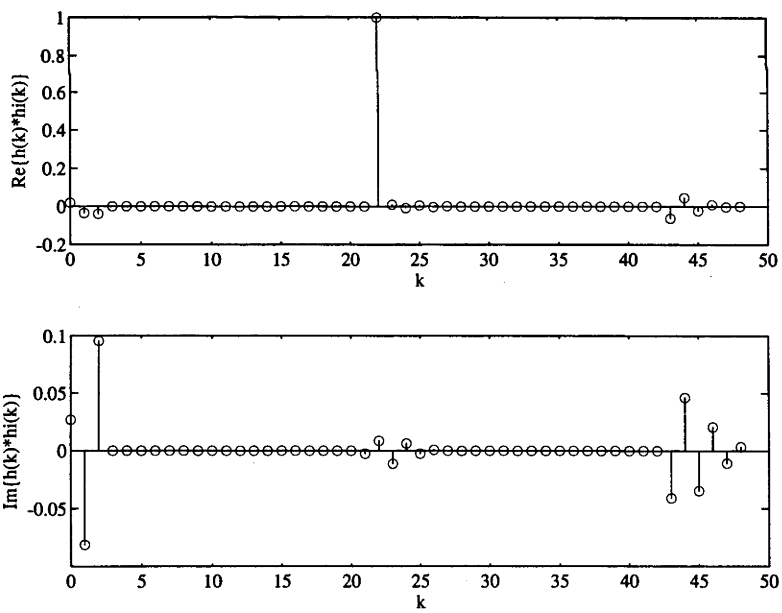


Figure 2-6: Convolution of the complex channel and truncated noncausal inverse of the channel of Figure 2-4.

Chapter 3

Approaches to Blind Equalisation

The notion of *admissibility* was introduced in Chapter 1, Section 1.1.3.2 as a highly desirable convergence property of blind equalisation algorithms. However, none of the practically implementable versions of the popular algorithms based on the *modified error signal* approach strictly possess this property. Nevertheless, MESA's currently enjoy the popularity of being the only class of algorithms computationally simple enough to be implemented on hardware for telecommunication applications. Therefore, further study towards the development of an *admissible* MESA is clearly warranted.

In this chapter, we identify the different approaches taken thus far within this class of algorithms and determine their main characteristics. We also highlight their inherent limitations in an attempt to deduce an alternative approach that leads to an admissible design. We describe in detail the class of MESA's in Section 3.1 and attempt to classify different approaches within this class according to various criteria including their distinguishing properties and/or limitations and their admissibility in Sections 3.2-3.4. The purpose of this general classification is to collate the major contributions concerned with the design and analysis of those blind equalisation algorithms that have appeared in the literature. In doing so, less desirable approaches are exposed and some are identified which might offer some benefit over the others. We present a summary of our results in Section 3.5. Finally, it should be noted that a portion of the original contributions found in this chapter is presented in a conference paper [23]. Additional contributions may be found in [23] which complements the work in this chapter.

3.1 Modified Error Signal Algorithms

We have seen in Chapter 1 that MESA's are characteristically *blind* because they do not utilise the actual *prediction error* quantity used by conventional training sequence-based equalisation algorithms. For example, the (nonblind) *least mean square (LMS)* algorithm adapts the equaliser parameters $\{\theta_i\}$ according to

$$\theta_i(k+1) = \theta_i(k) - \mu e_k x_{k-i}, \quad \forall\, i \tag{3.1.1}$$

where e_k is the *prediction error.* In place of the prediction error, MESA's use a blind *modified error signal.* In this section we consider the general form of a MESA and identify how various modified error signals are derived.

The essential element of any MESA is its *cost function* $\Psi(z_k)$ which we assume, without loss of generality, to be a *positive*, *even*, and *continuous* function of the equaliser output, $z_k \in \mathbb{C}$, that is,

$$\Psi(\cdot) : \mathbb{C} \to \mathbb{R}^+. \tag{3.1.2}$$

Furthermore, it is also assumed that

$$\Psi(z) \to \infty \text{ as } |z| \to \pm\infty. \tag{3.1.3}$$

The cost function itself is blind because it does not depend on explicit knowledge of the unknown channel input. In fact, to evaluate the cost requires only the equaliser output at time k, z_k, which is computed according to

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i x_{k-i}. \tag{3.1.4}$$

The equaliser parameter update is governed by a *stochastic gradient descent* of the cost. The ith parameter of the equaliser is updated according to:

$$\theta_i(k+1) = \theta_i(k) - \mu \frac{\partial}{\partial \theta_i} \Psi(z_k) \tag{3.1.5}$$

where μ is the update step-size parameter which is assumed sufficiently small for stability. Evaluating the partial derivative using the representation of z_k given by (3.1.4), we obtain

$$\theta_i(k+1) = \theta_i(k) - \mu \psi(z_k) x_{k-i} \tag{3.1.6}$$

where

$$\frac{\partial z_k}{\partial \theta_i} = x_{k-i} \tag{3.1.7}$$

and

$$\psi(z) = \Psi'(z). \tag{3.1.8}$$

Comparing the form of the blind parameter update in (3.1.6) with the nonblind prediction error based update recursion of (3.1.1), it can be seen that the gradient of the cost function $\psi(z_k)$ is in fact the *modified error signal* as it replaces the role of the prediction error.

Given a fixed equaliser parameter setting, the values taken by the cost function $\Psi(z_k)$ are stochastic in nature because the equaliser input and, hence its output z_k, is stochastic. The deterministic concept of a *mean cost function* or *surface* $\mathcal{J}(\theta)$ is more amenable for the analysis of the convergence behaviour of the algorithm:

$$\mathcal{J}(\theta) \triangleq \mathrm{E}\left[\Psi(z_k)\right]|_{z_k=\sum_{i=-\infty}^{\infty}\theta_i x_{k-i}} \tag{3.1.9}$$

where $\mathrm{E}[\cdot]$ denotes expectation over all possible symbol sequences $\{a_k\}$. Note that, as a result of the expectation operation, the mean cost function $\mathcal{J}(\cdot)$ is a deterministic function of the equaliser parameters $\{\theta_i\}$.

The parameter adaptation described by (3.1.6) can be interpreted as a gradient descent of a noisy estimate of the mean cost function $\mathcal{J}(\cdot)$ provided the step-size μ is small enough. On average, the equaliser parameters will converge to the *local stable minima* of the mean cost function. Therefore, determination of these minima is sufficient to characterise the convergence behaviour of the algorithm. The equilibria of the mean cost function are solutions in θ of

$$\frac{\partial}{\partial\theta_i}\mathcal{J}(\theta) = \mathrm{E}\left[\psi(z_k)x_{k-i}\right] = 0, \quad \forall\, i. \tag{3.1.10}$$

Determining whether the equilibria are stable minima or not can usually be determined by examining the Hessian $\boldsymbol{H}(\cdot)$ of $\mathcal{J}(\theta)$ [21], that is,

$$\boldsymbol{H}(\theta) \triangleq \left.\frac{\partial^2\mathrm{E}\left[\Psi(z_k)\right]}{\partial^2\theta(k)}\right|_{\theta(k)=\theta} = \mathrm{E}\left[\frac{\partial\psi(z_k)}{\partial\theta(k)}\right]\Bigg|_{\theta(k)=\theta} \tag{3.1.11}$$

where $\theta(k)$ denotes the equaliser parameter sequence $\{\theta_i\}$ at time k. Note that the commutativity of differentiation and expectation indicated by (3.1.11) is conditional upon $\Psi(\cdot)$ being smooth.

Given the convergence information that the mean cost function offers, the design of a MESA focuses on the design of a mean cost function that, *ideally*, possesses *stable global minima* which all correspond to zero-ISI equaliser parameter settings.

3.2 Admissibility

The term *admissibility* was introduced by [22, 23] as a desirable property of *practical* blind equalisation algorithms.

Definition 3.1 (Admissibility) *The "admissibility" of a blind equalisation algorithm is its ability to guarantee the adaptation of the equaliser parameters to the vicinity of a zero-ISI equaliser parameter setting from an arbitrary parameter initialisation.*

To explain what is meant by "vicinity of a zero-ISI equaliser parameter setting" we briefly recap the relevant notions from Chapter 2, Section 2.5. A *zero-ISI equaliser parameter setting* is one that produces a transversal equaliser output that does not contain any ISI. It is a possibly scaled and delayed version of the inverse of the channel impulse response and is of the form

$$\theta_i = \gamma \tilde{h}_{i-\Delta} \quad \forall\ i \tag{3.2.1}$$

where $\gamma \in \mathbb{C}$ and $\Delta \in \mathbb{Z}$ represent a possibly *unknown fixed gain* and *fixed delay* of the equaliser output relative to the channel input. The corresponding equaliser output with zero ISI is given by

$$z_k = \gamma a_{k-\Delta}, \quad \forall\ k, \tag{3.2.2}$$

where $\{a_k\}$ is the channel input symbol sequence. Note, however, that the zero-ISI equaliser parameter setting given by (3.2.1) generally implies that the transversal equaliser must have an *infinite* number of parameters because the impulse response of the channel inverse may be of infinite length. A practical transversal equaliser realisation must have a *finite* number of parameters and, therefore, may be only able to provide an *approximation* of a zero-ISI parameter setting. Thus, a practical *equalisation objective* is

$$z_k \approx \gamma a_{k-\Delta}, \quad \forall\ k \tag{3.2.3}$$

where the degree of approximation implied by "$\approx$" is such that the quantising function $\mathcal{Q}(\cdot)$ at the output of the equaliser is able to make correct decisions, *i.e.*, the equaliser output eye diagram is open. In the case where the channel inverse has an infinite-length impulse response, the stable global minima of the mean cost function $\mathcal{J}(\cdot)$ can at best only approximately achieve the zero-ISI equaliser parameter settings.

Therefore, by "vicinity of a zero-ISI equaliser parameter setting", it is meant that the finite set of equaliser parameters is adapted to a finite approximation of (3.2.1) that produces an *open-eye equaliser output* in the sense of (3.2.3).

3.3 Classification Criteria

In this section we present three criteria which we will use for our classification of the various approaches to blind equalisation. We will discuss the general implications of each criterion on the properties of MESA's. In our classification, we make the following assumptions:

1. Only blind equalisation algorithms suitable for *real (PAM)* transmission systems are considered to simplify our analysis. However, the qualitative properties do carry over to algorithms for complex QAM transmission systems.

2. A *linear transversal model* is assumed for our equaliser because it is most often used in practice. Its simple structure implies its subsequent ease of implementation.

The three criteria of classification we present here are:

1. Whether the mean cost surface $\mathcal{J}(\theta)$ is *unimodal* or *multimodal.*

2. Whether the equaliser is *infinitely* or *finitely* parametrised.

3. The *degree of symbol restoration* achieved at the output of the equaliser with an open-eye equaliser parameter setting.

In the following sections we provide a description of each criterion and discuss the broad impact that each may have on the *admissibility* of a MESA.

Unimodal/Multimodal Cost Surface

A mean cost surface $\mathcal{J}(\theta)$ is classified as either *unimodal* or *multimodal* depending on the number of minima it possesses. *Unimodal* generally implies a single global minimum with no local minima. Adopting a rather broad definition, we allow the global minimum to be realised at a *point* or a *compact set* in the space of equaliser parameters (e.g., the surface might be bowl shaped with a flat bottom).

From a gradient descent viewpoint, the convergence properties of the algorithm can be deduced from the identification of the global stable minima of the mean cost surface. Note that (3.1.3) implies divergence of the equaliser parameters to infinity is impossible. Here we take a convergence analysis to mean the identification and characterisation of the extrema of the mean cost surface rather than the establishment of a detailed picture of the convergence of the equaliser parameters to the extrema. In this way, our analysis is general and does not depend on the particular cost minimisation scheme employed.

Since a unimodal mean cost surface has a single global minimum, the convergence of the equaliser parameters of the associated algorithm does not depend on the initialisation of the parameters. In this sense, the analysis of the convergence properties of the algorithm is relatively simple. On the other hand, a *multimodal* mean cost surface is characterised by multiple isolated global stable minima. In this case, the convergence of the equaliser parameters to a particular stable minimum depends on the initialisation state of the parameters. This makes the analysis of the convergence properties of an algorithm adopting a multimodal mean cost function more difficult.

Infinite/Finite Equaliser Parametrisation

Closely tied to the type of mean cost surface is whether the linear transversal equaliser is *infinitely parametrised* or *finitely parametrised.* We have seen from (3.2.1) that a zero-ISI equaliser parameter setting is a scaled and delayed version of the stable channel inverse impulse response. The channel inverse impulse response may be of infinite duration and this implies that the zero-ISI transversal equaliser parameter setting requires an infinite number of parameters. Furthermore, the channel is possibly *mixed-phase* which implies that the stable impulse response of its inverse is in general *noncausal.* Therefore to account for these cases, a *double-sided infinite-length equaliser parametrisation* is assumed for the ideal realisation of a zero-ISI equaliser parameter setting, that is,

$$\{\theta_i\} = \{\ldots,\ \theta_{-2},\ \theta_{-1},\ \theta_0,\ \theta_1,\ \theta_2,\ \ldots\}\,. \tag{3.3.1}$$

However, a *practical* transversal equaliser cannot possess an infinite number of parameters nor can it be noncausal. A causal finitely parametrised equaliser can only *approximate* a zero-ISI parameter setting when the channel inverse impulse response is of infinite duration. We assume here the following finite equaliser parametrisation shown in delay operator (z^{-1}) form:

$$z^{-N} \sum_{i=-N}^{N} \theta_i z^{-i}. \tag{3.3.2}$$

It is assumed that an adequate approximation is achieved when N is large enough. The parametrisation described by (3.3.2) implies a double-sided truncation of the ideal double-sided infinite-length equaliser parametrisation followed by the application of a delay of z^{-N} to make them causal. We remark that there is no strict requirement that the truncation be symmetric about the θ_0 term. Here, we adopt the symmetric form for convenience.

Although an infinitely parametrised equaliser is not practical, we include this case in our classification for comparison with the finite equaliser parametrisation case. The comparison is necessary because, as we shall see later, some algorithms deemed *admissible* assuming an *infinite* equaliser parametrisation have in fact been shown to be *inadmissible* when the number of equaliser parameters is made *finite* (and arbitrarily large) [20, 21, 13].

Degree of Symbol Restoration

The last criterion we consider is the *degree of symbol restoration* achieved at the output of the equaliser with an open-eye equaliser parameter setting. We may roughly divide the algorithms into those which seek to restore the symbol constellation or alphabet at the output of the equaliser as in (3.2.3) with $\gamma = \pm 1$ and those which seek to do so in some weaker sense (*e.g.*, γ arbitrary and nonzero).

3.3.1 Alphabet Restoration

This *alphabet restoration criterion* has its theoretical basis in a theorem found in [19] for the case of a real PAM alphabet. Stated simply (the result for a complex QAM constellation is expected to be similar):

Theorem 3.1 Alphabet Restoration *Let an infinitely parametrised linear transversal equaliser $\{\theta_i\}$ be placed in cascade following a linear stable channel $\{h_i\}$. The channel input $\{a_k\}$ consists of symbols from a symmetric PAM alphabet $\mathcal{A}$. The output of the linear channel $\{x_k\}$ is given by*

$$x_k = \sum_{i=0}^{\infty} h_i a_{k-i} \quad \forall\ k. \tag{3.3.3}$$

The corresponding output of the equaliser $\{z_k\}$ is computed according to

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i x_{k-i} \quad \forall\ k. \tag{3.3.4}$$

Let $p_z(\cdot)$ denote the distribution of the equaliser output $\{z_k\}$ and $p_a(\cdot)$ denote the distribution of the channel input $\{a_k\}$. Provided $p_a(\cdot)$ is not Gaussian, $p_z(\cdot) = p_a(\cdot)$ implies that

$$z_k = \pm a_{k-\Delta} \quad \forall\ k \tag{3.3.5}$$

for some $\Delta \in \mathbb{Z}$.

Note that (3.3.5) is equivalent to the zero-ISI equaliser output of (3.2.2) with the magnitude of the unknown gain factor $|\gamma| = 1$. The corresponding zero-ISI equaliser parameter setting is given by (3.2.1) for $|\gamma| = 1$, namely,

$$\theta_i = \pm \widetilde{h}_{i-\Delta}, \quad \forall\ i. \tag{3.3.6}$$

Note that (3.3.6) means that alphabet restoration criterion implies complete identification of the impulse response of the *channel inverse* $\{\widetilde{h}_i\}$ modulo an unknown delay Δ.

Therefore to create an admissible algorithm it is sufficient to drive $\{z_k\}$ to have the same first-order statistics as the (known) symbol constellation. For example, for a binary PAM alphabet, once $\{z_k\}$ becomes binary then (3.3.5) holds for some Δ.

This criterion is adopted by most of the well-known MESA's including the *Sato* [6], *Benveniste-Goursat-Ruget (BGR)* [19], *Godard* [7], and the *Constant Modulus Algorithm (CMA)* [11]. They attempt to restore the symbol constellation by restoring a particular stationary property of the channel input symbols at the output of the equaliser.

3.3.2 Equalisation without Gain Identification

The *equalisation without gain identification (EWGI)* criterion [28, 29, 24, 26] implies a weaker identification of the channel inverse in the sense that it permits an *arbitrary* value of γ in (3.2.2).

For the real case, satisfaction of the EWGI criterion implies

$$z_k = \gamma_\Delta a_{k-\Delta}, \quad \forall\ k \tag{3.3.7}$$

where $\gamma_\Delta \neq 0 \in \mathbb{R}$ and Δ represent an unknown fixed gain and delay of the zero-ISI equaliser output relative to the channel input. The corresponding equaliser parameter setting is

$$\theta_i = \gamma_\Delta \widetilde{h}_{i-\Delta}, \quad \forall\ i. \tag{3.3.8}$$

The idea is to relax the identification of γ_Δ and, thus, we do not expect to restore the alphabet perfectly but rather restore it only up to a nonzero gain factor. The indicated dependence of the gain γ on the delay Δ is a fundamental property of the EWGI criterion which we will illustrate later by our example. We also make the important remark that the relaxing of the requirement to identify the gain incurs little consequence because it can be recovered with a separate automatic gain control stage following the equaliser [24, 26].

An algorithm based on the EWGI criterion represents a departure from the usual *alphabet restoration* philosophy of blind equalisation. The alphabet restoration criterion implies that all the parameters of the equaliser $\{\theta_i\}$ must be fully adjustable in order to identify the inverse of the channel in the sense of (3.3.6). On the other hand, an EWGI algorithm employs a *linear constraint* on the equaliser parameters making it generally impossible for them to identify the correct gain of the channel inverse sequence $\{\widetilde{h}_i\}$.

We now give a simple example which illustrates the fundamental properties of an EWGI algorithm. This will be followed by some more general observations. Consider the case where we choose as in [29, 24] the single parameter constraint $\theta_0 = 1$. Suppose we have a single-pole channel of the form

$$H(z^{-1}) = \frac{1}{1 + 2z^{-1}} \tag{3.3.9}$$

where we $H(z^{-1})$ denotes the delay operator form of the channel impulse response $\{h_k\}$ and z^{-1} is the unit delay operator. Without a constraint, there exists an infinite number of possible zero-ISI equaliser parameter settings that satisfy (3.3.7). For example, consider all the shifted versions of the reciprocal of (3.3.9), *i.e.*,

$$\theta(z^{-1}) = (1 + 2z^{-1})z^{-\Delta}, \quad \Delta \in \mathbb{Z} \tag{3.3.10}$$

where $\theta(z^{-1})$ denotes the Z-transform of the equaliser parameters $\{\theta_i\}$. However, *under the constraint*, there exists only two zero-ISI equaliser parametrisations, namely,

$$\theta^{(1)}(z^{-1}) = 1 + 2z^{-1} \tag{3.3.11}$$

$$\theta^{(2)}(z^{-1}) = \frac{1}{2}z + 1. \tag{3.3.12}$$

Note that these parametrisations are of the form given by (3.3.8) and are constructed by considering all delayed versions of the channel inverse, $\widetilde{H}(z^{-1}) = 1 + 2z^{-1}$, for which there exists a nonzero zeroth element of the delayed sequence and normalising with respect to that zeroth element. The constructions take the form

$$\theta_i = \frac{\widetilde{h}_{i-\Delta}}{\widetilde{h}_{-\Delta}} \quad \forall\, i \tag{3.3.13}$$

for all Δ such that $\widetilde{h}_{-\Delta} \neq 0$.

The corresponding equaliser outputs of (3.3.11) and (3.3.12) are

$$z_k^{(1)} = a_k \quad \forall\, k \tag{3.3.14}$$

$$z_k^{(2)} = \frac{1}{2}a_{k+1} \quad \forall\, k. \tag{3.3.15}$$

The first parametrisation $\theta^{(1)}$ corresponds to the gain $\gamma_\Delta = 1$ where $\Delta = 0$ whereas the second $\theta^{(2)}$ corresponds to the gain $\gamma_\Delta = 1/2$ where $\Delta = -1$. This brings out a *fundamental* characteristic of the EWGI approach, that is, the *dependence of the gain* γ *on the delay* Δ. It is important to note from (3.3.13) that the delay Δ corresponds to the delay of the channel inverse adopted by the equaliser parameters. Furthermore, the nature of the dependence of the gain on the delay is determined by both the linear constraint and the impulse response of the channel inverse $\{\widetilde{h}_i\}$. This is illustrated more clearly when we consider a *general linear constraint* of the form

$$\sum_{i=-\infty}^{\infty} c_i \theta_i = 1 \tag{3.3.16}$$

where the constraint sequence $\{c_i\}$ is arbitrary. Substituting the general form of the zero-ISI equaliser parameter setting given by (3.3.8) into (3.3.16) and solving for the gain γ_Δ, we obtain

$$\gamma_\Delta = \frac{1}{\sum_{i=-\infty}^{\infty} c_i \widetilde{h}_{i-\Delta}}. \tag{3.3.17}$$

Equation (3.3.17) indicates explicitly how the delay dependency of the gain is induced by the linear constraint $\{c_i\}$ and the channel inverse $\{\widetilde{h}_i\}$.

Another basic property of an EWGI algorithm is that the cost function is designed to be a *monotonically increasing* function of the magnitude of the delay-dependent gain $|\gamma_\Delta|$. Assuming such a cost in this example, we can conclude that the value of the cost corresponding to the second equaliser parameter setting $\theta^{(2)}$ will be lower than that corresponding to the first because its gain γ_{-1} is lower. We make the following additional remarks.

Remarks

1. It is the imposition of a linear constraint on the equaliser parameters $\{\theta_i\}$ that effectively destroys the ability of a EWGI algorithm to identify the gain γ of the zero-ISI equaliser output relative to the channel input symbols. However, a significant outcome is that the gain γ_Δ becomes a function of the delay Δ. This is made explicit by (3.3.17) where it can be seen that γ_Δ is a function of the linear constraint sequence $\{c_i\}$ and the delayed channel inverse impulse response $\{\widetilde{h}_{i-\Delta}\}$.

2. Given the two previous properties mentioned, the cost function is also a function of the delay Δ and thereby has the potential to distinguish between different zero-ISI equaliser parametrisations corresponding to different delays. Consider the case where the linear constraint and channel induce a dependence of the gain γ_Δ on the delay Δ such that there exists a *unique* value of Δ that minimises γ_Δ. Then it is possible for the cost function to be uniquely minimised by the zero-ISI equaliser parameter setting corresponding to that Δ.

3. In contrast, the *alphabet restoration* criterion corresponds to the case where $\gamma_\Delta = 1, \forall\, \Delta$ and hence the cost function will exhibit the *same* value for all corresponding zero-ISI equaliser parameter settings.

So far we have seen that a cost function designed around the EWGI philosophy has the potential to distinguish between different zero-ISI equaliser parameter settings corresponding to different delays of the equalised output relative to the channel input. In the ideal case, a particular zero-ISI equaliser parameter setting corresponding to a particular delay can *uniquely* minimise the cost. However, this property hinges on a dependence of the gain γ_Δ on the delay Δ where the gain is uniquely minimised by a particular delay. Therefore, it is possible that a *particular* combination of the linear constraint and channel inverse may induce a nonunique dependence of the gain on the delay. In particular, it may be possible for the gain γ_Δ to be minimised nonuniquely by more than one delay Δ.

We now give a theorem on a *nonuniqueness phenomenon* that is an artefact of the EWGI philosophy introduced by pathological or *nongeneric* combinations of the channel inverse and linear parameter constraint. The nonuniqueness mechanism is illustrated in the proof of the theorem.

Theorem 3.2 Nonuniqueness *Consider a linear stable possibly mixed-phase channel with impulse response $\{h_k\}$ followed by a linear transversal equaliser with parameters $\{\theta_i\}$ which are subject to a linear constraint of the form*

$$\sum_{i=-\infty}^{\infty} c_i \theta_i = 1 \tag{3.3.18}$$

where $\{c_i\}$ is the constraint sequence. The channel input symbol sequence is $\{a_k\}$ and the equaliser output is $\{z_k\}$. Suppose that a cost function is a monotonically increasing function of the magnitude of the gain of the equaliser output relative to the channel input. Also, let this cost be globally minimised by at least one zero-ISI equaliser parameter setting. Then, there exists channels $\{h_k\}$ such that the global minimisation of the cost is realised by at least two distinct zero-ISI equaliser parameter settings.

Proof: A general proof can be developed as a straightforward extension of the specific construction to follow. Consider the simple linear constraint of $\theta_0 = 1$ and the channel

$$H(z^{-1}) = \frac{1}{1 + z^{-1} - z^{-2}} \tag{3.3.19}$$

where $H(z^{-1})$ denotes the delay operator form of the channel impulse response $\{h_i\}$. There are only three possible zero-ISI equaliser parameter settings that conform to the constraint, namely,

$$\theta^{(1)}(z^{-1}) = 1 + z^{-1} - z^{-2} \tag{3.3.20}$$

$$\theta^{(2)}(z^{-1}) = z + 1 - z^{-1} \tag{3.3.21}$$

$$\theta^{(3)}(z^{-1}) = z^2 + z - 1. \tag{3.3.22}$$

The corresponding equaliser outputs are

$$z_k^{(1)} = a_k \quad \forall\, k \tag{3.3.23}$$

$$z_k^{(2)} = a_{k+1} \quad \forall\, k \tag{3.3.24}$$

$$z_k^{(3)} = -a_{k+2} \quad \forall\, k. \tag{3.3.25}$$

The first parametrisation $\theta^{(1)}$ corresponds to the gain $\gamma_\Delta = 1$ where $\Delta = 0$ whereas the second $\theta^{(2)}$ also corresponds to the gain $\gamma_\Delta = 1$ where $\Delta = -1$. The third parametrisation $\theta^{(3)}$ corresponds to the gain $\gamma_\Delta = -1$ where $\Delta = -2$. Because the magnitudes of the gains are identical, the gain-dependent cost will be the same for all three distinct equaliser parameter settings. Since we have assumed that the cost is globally minimised by at least one open-eye equaliser parametrisation then all three parametrisations achieve the same minimum cost value. ∎

Remarks

1. The proof uses a particular constraint of $\theta_0 = 1$ to illustrate the concepts. However, it can be easily generalised to a general constraint of the form (3.3.16). Therefore, the result is applicable to a particular EWGI algorithm adopting a general linear parameter constraint of that form that we examine in Chapter 5.

2. From the proof, we see that *nonuniqueness* is attributed to a *particular* (pathological) pairing of the constraint and channel (or channel inverse) and does not depend on the particular type of cost used. In the case illustrated, the magnitude of the gain $|\gamma_\Delta|$ is identical for three different values of Δ that correspond to zero-ISI equaliser parametrisations. Consequently, no single zero-ISI equaliser parameter setting can uniquely achieve the minimisation of the cost.

3. If the cost function is assumed *unimodal*, then the nonunique cost-minimising zero-ISI equaliser parameter settings must all lie on some connecting *manifold* (constituting a flat bottom of the cost) in the equaliser parameter space. Generally, this manifold may contain closed-eye equaliser parameter settings and this may be a disadvantage. However, if the cost is *multimodal*, then it is possible for each nonunique cost minimising zero-ISI equaliser parameter setting to lie in isolated global minima of the cost surface.

3.4 Results

Permutations of the three classification criteria of the previous section (*i.e.*, unimodal or multimodal cost surface, infinite or finite equaliser parametrisation, degree of symbol restoration) yields eight potential classes of blind algorithms that can be assessed according to their *admissibility*. They are:

1. Infinite equaliser parametrisation

   (i) alphabet restoration, unimodal cost

   (ii) alphabet restoration, multimodal cost

   (iii) EWGI, unimodal cost

   (iv) EWGI, multimodal cost

2. Finite equaliser parametrisation

   (i) alphabet restoration, unimodal cost

   (ii) alphabet restoration, multimodal cost

   (iii) EWGI, unimodal cost

   (iv) EWGI, multimodal cost

However, the classes of multimodal algorithms adopting the EWGI criterion under both infinite and finite equaliser parametrisations are not considered here. A discussion of each of the remaining six classes follows.

3.4.1 Infinite Parametrisation Case

Alphabet Restoration with Unimodal Cost

Although we are not aware of any practical algorithms in this class at the moment we can deduce their properties. The following theorem in fact brands this class as *inadmissible.*

Theorem 3.3 Inadmissibility Result *The simultaneous specification of unimodality and perfect alphabet restoration are incompatible, i.e., cannot lead to an admissible algorithm.*

Proof: For such an algorithm to exist, we must have at least one *ideal equaliser parameter setting* $\{\theta_i^*\}$ that realises the global minimum of the unimodal cost. For perfect alphabet restoration, $\{\theta_i^*\}$ takes the form

$$\theta_i^* = \gamma^* \widetilde{h}_{i-\Delta^*} \quad \forall\, i \tag{3.4.1}$$

where $\gamma^* = \pm 1$ and $\{\widetilde{h}_i\}$ denotes the channel inverse sequence and Δ^* represents a particular fixed delay. The corresponding equaliser output is given by

$$\begin{aligned} z_k^* &= h_k * \theta_k^* * a_k = h_k * \gamma^* \widetilde{h}_{k-\Delta^*} * a_k \\ &= \gamma^* a_{k-\Delta^*}. \end{aligned} \tag{3.4.2}$$

Now let $\{\theta_i'\}$ be another ideal equaliser parameter setting, that is,

$$\theta_i' = \gamma' \widetilde{h}_{i-\Delta'} \quad \forall\, i \tag{3.4.3}$$

where $\gamma' = \pm 1$ and $\Delta' \neq \Delta$. The corresponding equaliser output is given by

$$z_k' = \gamma' a_{k-\Delta'}. \tag{3.4.4}$$

However, because a blind cost is *even* and is inherently *insensitive* to delays in its argument (see Section 3.1), we have

$$\begin{aligned} \Psi(z_k') &= \Psi(\gamma' a_{k-\Delta'}) = \Psi(\gamma^* a_{k-\Delta^*}) \\ &= \Psi(z_k^*) \end{aligned} \tag{3.4.5}$$

or in expected value form

$$\mathcal{J}(\theta') = \mathcal{J}(\theta^*). \tag{3.4.6}$$

Therefore, we conclude that if there exists at least one zero-ISI equaliser parameter setting that globally minimises the cost, then all zero-ISI equaliser parameter settings must minimise the cost. Now, by our notion of *unimodality*, all of the zero-ISI equaliser parameter settings must belong to some connecting manifold $\mathcal{M}$ which globally minimises the mean cost $\mathcal{J}(\cdot)$.

Generally, other parameter settings within $\mathcal{M}$ are not necessarily zero-ISI equaliser parameter settings, *i.e.*, do not satisfy (3.3.7). For example, if the set $\mathcal{M}$ is *convex*, then any convex combination of zero-ISI parameter settings is an element of $\mathcal{M}$. Consider the *convex hull* of the three ideal parameter settings $\theta^{(j)} = \gamma^{(j)}\widetilde{h}_{i-\Delta^{(j)}} \in \mathcal{M}$, $j = 1,\ 2,\ 3$.

$$\begin{aligned}\theta(\lambda)_i &= \sum_{j=1}^{3} \lambda_j \theta^{(j)} \\ &= \sum_{j=1}^{3} \lambda_j \gamma^{(j)} \widetilde{h}_{i-\Delta^{(j)}} \end{aligned} \tag{3.4.7}$$

where

$$\sum_{j=1}^{3} \lambda_j = 1. \tag{3.4.8}$$

The corresponding equaliser output is given by

$$\begin{aligned} z_k(\lambda) &= h_k * \left[\sum_{j=1}^{3} \lambda_j \gamma^{(j)} \widetilde{h}_{i-\Delta^{(j)}} \right] * a_k \\ &= \sum_{j=1}^{3} \lambda_j \gamma^{(j)} h_k * \widetilde{h}_{k-\Delta^{(j)}} * a_k \\ &= \sum_{j=1}^{3} \lambda_j \gamma^{(j)} a_{k-\Delta^{(j)}}. \end{aligned} \tag{3.4.9}$$

The form of (3.4.9) indicates that, in general, we can have the situation where no term of the sum dominates the rest of the sum, therefore, indicating a closed-eye equaliser parameter setting. ∎

Alphabet Restoration with Multimodal Cost

We have seen that the previous class combining the equalisation criteria of alphabet restoration with a unimodal cost is inadmissible. Here, we investigate the class with the same equalisation criterion but have replaced the requirement of a unimodal cost with a multimodal one.

A large number of MESA's fall under this category [6, 7, 19] including the well-known CMA [11]. Given that the mean cost surface is multimodal, then the algorithm would be *admissible* if all local minima are (i) global minima, and (ii) correspond precisely to zero-ISI equaliser parameter settings for the various delays Δ in (2.5.4). That is, for every delay, Δ, in (2.5.4) with $\gamma = \pm 1$, there exists a zero-ISI equaliser parameter setting corresponding to an *isolated* global minimum of the cost.

Using the assumption that the equaliser parametrisation is double-sided and infinite-length, a class of alphabet restoration-multimodal cost algorithms was proposed by Godard [7] with costs of the form

$$\Psi_p(z) \triangleq \frac{1}{2p}\left(|z|^p - r_p\right)^2 \tag{3.4.10}$$

where

$$r_p = \frac{\mathrm{E}\left[|a_k|^{2p}\right]}{\mathrm{E}\left[|a_k|^p\right]}, \quad p \geq 2. \tag{3.4.11}$$

The CMA is in fact a specific case of the Godard class of algorithms for $p = 2$. The *admissibility* of this class has so far been established for $p = 2$ only [7]. However, admissibility of this class cannot be assumed in the case of finite equaliser parametrisation. We shall consider this case in Section 3.4.2.

EWGI with Unimodal Cost

Given our discussion of EWGI in Section 3.3.2, we need only augment the discussion there by assuming that the cost function is unimodal. There we have seen that it is possible for an EWGI cost to be *uniquely* minimised by a particular zero-ISI equaliser parameter setting. If the cost is unimodal, then we expect a gradient descent-based cost minimisation scheme to exhibit global convergence to the particular zero-ISI equaliser parameter setting irrespective of the state of the initialisation of the parameters. Therefore, the combination of the EWGI criterion and unimodal cost has the potential to yield an *admissible* algorithm.

The *nonuniqueness phenomenon* is an inherent artefact of EWGI where a cost is nonuniquely minimised by two or more zero-ISI equaliser parameter settings (see Theorem 3.2). In the case where the cost is unimodal, this simply means that those zero-ISI equaliser parameter settings must belong to some connecting manifold in the equaliser parameter space which may also contain equaliser parameter settings that do not necessarily correspond to zero-ISI settings.

An example of an algorithm in this class is one first proposed in [28] and later theoretically proven to be admissible (in the generic case) in [29, 24, 26]. The algorithm adopts a unimodal cost that is *convex* in the equaliser parameters and is of the form

$$\Psi(z_k) = \max_{\{a_k\}} |z_k| \tag{3.4.12}$$

subject to the now familiar constraint $\theta_0 = 1$. The fact that the cost is *convex* does not alter our discussion. However, it becomes important when we consider this class of algorithms in the finite equaliser parametrisation case next.

3.4.2 Finite Parametrisation Case

For the purposes of the discussions to follow we assume a practical finite equaliser parametrisation of the form given in (3.3.2) where a parametrisation parameter N gives a equaliser parametrisation consisting of $2N + 1$ parameters. We note here that, as a result of the finite parametrisation, the equaliser parameters can at best achieve an *open-eye equaliser parameter setting* as opposed to the ideal case of a zero-ISI setting.

Alphabet Restoration with Unimodal Cost

We simply conclude that this class is *inadmissible* in light of its *inadmissibility* in the infinite parametrisation case.

Alphabet Restoration with Multimodal Cost

Here, we report from recent literature that assesses the *admissibility* of many of the well-known algorithms in this class under finite equaliser parametrisation. In general, it can be established using a simple example that finitely parametrising an equaliser introduces stable local minima into the multimodal cost that do not necessarily correspond to open-eye equaliser parameter settings. This *local minima problem* introduces the possibility of the equaliser adapting its parameters to one of these settings and failing to open the eye diagram [20, 21, 13]. This fact renders these algorithms *inadmissible.*

Some specific results follow from [20, 21, 13, 30]:

1. Even when the channel is *minimum-phase* it can be shown that the effect of finitely parametrising the equaliser is to introduce additional stable local minima that do not correspond to open-eye equaliser parameter settings. These minima exist for *all* finite N and do *not* diminish in depth (as measured by certain eigenvalues of the Hessian of the cost) with increasing N.

2. Even when the finitely parametrised equaliser can attain perfect equalisation (for certain autoregressive channels) these local minima still exist [21].

3. The demonstration of the inadmissibility of these algorithms emphasises the importance of *parameter initialisation* [7] to avoid convergence to the spurious local minima introduced under finite equaliser parametrisation.

4. For the BGR algorithms it has also been established that the popular center-spike initialisation can fail on a number of channels [30]. Analogous results for the Godard algorithm are unknown.

EWGI with Unimodal Cost

It is easy to construct an example illustrating the inadmissibility of an EWGI algorithm with a specific unimodal cost under finite equaliser parametrisation. Therefore, the *entire* class cannot be established as admissible. However, a subset of these algorithms which adopt unimodal costs that are additionally *convex* in the equaliser parameters has been considered. It can be shown that the minimum of a linearly constrained convex cost can identify a zero-ISI equaliser parameter setting to an arbitrary degree of accuracy with N made suitably large [24]. This result exploits the fact that convex functions remain convex under linear constraints on its parameters. Also, the truncation of an infinitely parametrised equaliser to a finitely parametrised one is equivalent to applying a series of trivial linear parameter constraints on the infinitely parametrised equaliser. Therefore, finitely parametrising the equaliser does not destroy the convexity property of the cost [29, 24, 26]. It can then be concluded that an algorithm based on a linearly-constrained convex cost function and deemed admissible assuming an infinite equaliser parametrisation will remain admissible when there is a sufficient though finite number of equaliser parameters. Therefore, such an algorithm does not suffer from the *local minima problem* experienced by those algorithms adopting nonconvex multimodal cost functions under finite parametrisation. To support this claim, it was demonstrated in [24] that the algorithm using the convex cost and constraint given by (3.4.12) is *admissible* under a finite equaliser parametrisation [24]. That is, the equaliser parameters are adapted to an approximation of a zero-ISI equaliser parameter setting with an increasing degree of accuracy as N is made large. Finally, note that the *nonuniqueness phenomenon* is still possible given that its underlying mechanism is independent of whether the equaliser is infinitely or finitely parametrised.

3.5 Summary

We have described an important class of blind equalisation algorithms known as *modified error signal algorithms* or *MESA*'s. From its ranks come most of the practical algorithms that we have seen to date in the literature and in practice such as the *Sato*, *BGR*, *Godard*, and *CMA* algorithms. Within this large class, we identified six categories of algorithms. The six categories were formed from permutations of three criteria, namely, whether a unimodal or multimodal cost function is adopted, whether an infinite or finite equaliser parametrisation is assumed in the theoretical development, and, finally, whether the alphabet restoration or EWGI criterion of symbol restoration is sought by the algorithm. We recalled a highly desirable property of MESA's known as *admissibility* and then based our evaluation of each subclass on this property.

Highlighting our findings, we developed results to indicate the inherent *inadmissibility* of the subclass of *alphabet restoration* algorithms adopting *unimodal* cost functions assuming both infinite and finite equaliser parametrisations. Most of the popular practical algorithms such as the *Sato*, *BGR*, *Godard*, and *CMA* algorithms were categorised under the subclass of *alphabet restoration* algorithms adopting a *multimodal* cost function. We reported from the literature pertaining to this class that most of the algorithms have been shown to be admissible assuming an *infinite* equaliser parametrisation. However, recent literature has unveiled the fact that, in the absence of a special parameter initialisation strategy, all of these algorithms possess the potential to converge to a closed-eye equaliser parameter setting when the number of equaliser parameters is made *finite*. This degenerate behaviour arises because the finite parametrisation introduces stable local minima as artefacts that do not correspond to open-eye equaliser parameter settings. Due to this *local minima problem* under finite parametrisation, it can be concluded that all *practical* versions of these algorithms are *inadmissible*. However, rather than being an overly pessimistic result, this highlights the need to develop special equaliser parameter initialisation schemes that can prevent adaptation to the undesirable local minima.

We described a relatively recent alternative to the alphabet restoration criterion known as the *equalisation without gain criterion (EWGI)*. A weaker criterion than the former, we demonstrated how it arises naturally out of the idea of linearly constraining the equaliser parameters. We also developed the result that algorithms adopting this criterion and a unimodal cost have the potential to be *admissible*. This is tied to the ability of the cost to be *uniquely* minimised by a particular zero-ISI equaliser parameter setting unlike in the class of alphabet restoration algorithms using unimodal costs. We drew attention to an algorithm for real PAM systems [29, 24, 26] based on a *convex*, hence, unimodal cost with a simple parameter constraint that was shown to be *admissible*. Furthermore, we reported from that work that using *convex* cost functions have the added advantage that they remain convex under a finite equaliser parametrisation or any other linear constraints on the parameters. Therefore, it is impossible for these costs to suffer from the *local minima problem*.

We also elaborated on a nongeneric artefact of linearly constraining the equaliser parameters to meet the EWGI criterion. The *nonuniqueness* phenomenon was identified as arising out of specific (nongeneric) combinations of the channel and parameter constraint and having the potential in the case of a unimodal cost to lead to ill-convergence. However, due to its relatively low probability of occurrence, it was not considered a major limitation. This phenomenon will be seen in a more general and intuitive setting later in Chapter 5 where we provide a geometric interpretation of the mechanisms responsible. Possible methods to circumvent the nonuniqueness phenomenon when it does occur may involve the application of a dynamic constraint or special nonlinear constraint as proposed in Chapter 6.

Finally, ranking nonuniqueness as a minor artefact, we conclude from our survey that the subclass of algorithms adopting the EWGI criterion and a *convex* cost seems to offer the best hope of yielding an *admissible* design that also retains its admissibility under practical finite equaliser parametrisations. This property makes this subclass considerably amenable to analysis. That is, it allows the *ideal* convergence properties of the algorithm to be analysed relatively easily in the infinite equaliser parameter space with the confidence that these properties can be approximated to an arbitrary degree of accuracy with a finite equaliser parametrisation. Analysis is also facilitated because convex costs result in *global* equaliser parameter convergence under a gradient descent scheme irrespective of the initialisation of the parameters.

Table 3.1 and Table 3.2 summarises the classification and assessment of blind algorithms in the infinite and finite parametrisation cases.

Table 3.1: Admissibility in the infinite parametrisation case.

Infinite Parametrisation	*Unimodal*	*Multimodal*
Alphabet Restoration	**Inadmissible:** Undesirable manifold of equaliser parameter settings that realises the global minimum of the cost	**Admissible:** [7]
Equalisation without Gain Identification	**Admissible:** *Nonuniqueness phenomenon* observed for specific channels. [28, 29, 24, 26]	—

Table 3.2: Admissibility in the finite parametrisation case.

Finite Parametrisation	*Unimodal*	*Multimodal*
Alphabet Restoration	**Inadmissible:** As in case of infinite parametrisation	**Inadmissible:** Even for sufficiently parametrised equalisers [30, 21, 31]
Equalisation without Gain Identification	**Admissible:** For convex functions, same as in case of infinite parametrisation	—

Chapter 4

Towards a Globally Convergent Algorithm

Drawing on the observations from the survey of practical algorithms in Chapter 3 one relatively new yet promising class of *modified error signal algorithms (MESA's)* has emerged as one that may provide theoretically justified *admissible* algorithms, namely, those algorithms which adopt the *equalisation without gain identification (EWGI)* criterion (Chapter 3, Section 3.3.2) and a *convex* cost function. This chapter considers successive generalisations of an EWGI algorithm based on a convex cost and single-parameter constraint originally proposed for real PAM systems by [29, 24, 26]. Our generalisations are intended to yield a design that is suitable for complex QAM systems. These generalisations attempt to illustrate that the subsequent development of our admissible algorithm in Chapter 5 is not based on ad-hoc methods, but rather a systematic approach that allows the algorithm to be analysed.

We begin, in Section 4.1, by raising the issues surrounding the design of a convex cost and linear equaliser parameter constraint that is suitable for QAM systems and adheres to the EWGI philosophy. In Section 4.2, we focus on the algorithm of [29, 24, 26] that was shown to be admissible for real PAM systems. Using simpler methods of analysis than those used by [29, 24, 26], we give insight into how the algorithm fulfills the desired features of the EWGI approach and consequently is admissible for the real case. Also, we give a trivial generalisation of the cost for QAM systems and highlight the reasons for its inadmissibility. In Section 4.3, we investigate another generalisation of the cost and constraint which we discover to be more suitable though applicable to a restricted class of QAM constellations. Finally, we provide a summary in Section 4.4.

4.1 Algorithm Design Philosophy

Here we discuss the philosophy behind the EWGI approach and highlight the issues that must be considered for the design of a suitable cost and constraint.

In formalising why we desire a cost function that is *convex* in the equaliser parameters we list the following reasons:

1. A convex cost function is inherently unimodal with a global minimum (in general). A local characterisation of the minimum applies globally. This facilitates the convergence analysis as, qualitatively, the convergence behaviour is independent of the initialisation of the equaliser parameters.

2. A convex cost remains convex under one or a multiple of *linear* constraints on the equaliser parameters. Furthermore, finite parametrisation of an equaliser is equivalent to multiple simple linear constraints on an infinitely parametrised equaliser and, therefore, does not destroy the convexity property of the cost.

3. There is a graceful degradation in the ability of the minimum of the cost to identify a zero-ISI equaliser parameter setting as the number of equaliser parameters is decreased from infinity to some finite number [24].

4. It is mathematically convenient to adopt a double-sided infinite-length parametrisation of the linear transversal equaliser because it permits the exact modeling of the double-sided inverse of a possibly mixed-phase channel $\{\widetilde{h}_i\}$, *i.e.*, zero-ISI equaliser parameter settings. However, any practical implementation must have a finite number of parameters. The use of a convex cost function thus allows the *analysis* of the convergence properties of the algorithm assuming the convenient infinite-length parametrisation with confidence that those properties will be retained to an arbitrary level of accuracy under a finite-length parametrisation.

The finite-length equaliser parametrisation we assume here is shown in delay operator (z^{-1}) form:

$$z^{-N} \sum_{i=-N}^{N} \theta_i z^{-i} \tag{4.1.1}$$

where $\theta_i \in \mathbb{C} \; \forall \; i$, and it is assumed that an adequate approximation of a zero-ISI equaliser parameter setting is achieved when N is large enough.

Given that a convex cost remains convex under a *linear* constraint on the parameters we adopt the following linear parameter constraint to achieve the EWGI objective.

$$\sum_{i=-\infty}^{\infty} \operatorname{Re}\{\bar{c}_{-i}\theta_i\} = 1 \tag{4.1.2}$$

where the constraint sequence $\{c_i\}$ consists of complex elements, $c_i \in \mathbb{C} \; \forall \; i$, and $\bar{c}_i$ denotes the complex conjugate of c_i. Equation (4.1.2) can be expanded to show that the constraint is a linear one in terms of the real and imaginary components of the equaliser parameters, namely,

$$\begin{aligned} \sum_{i=-\infty}^{\infty} \operatorname{Re}\{\bar{c}_{-i}\theta_i\} &= \sum_{i=-\infty}^{\infty} \operatorname{Re}\{\bar{c}_{-i}\} \operatorname{Re}\{\theta_i\} - \operatorname{Im}\{\bar{c}_{-i}\} \operatorname{Im}\{\theta_i\} \\ &= \sum_{i=-\infty}^{\infty} \operatorname{Re}\{c_{-i}\} \operatorname{Re}\{\theta_i\} + \operatorname{Im}\{c_{-i}\} \operatorname{Im}\{\theta_i\} = 1. \end{aligned} \tag{4.1.3}$$

Note that the form of the linear parameter constraint in (4.1.3) is applicable to the case where the equaliser parameters $\{\theta_i\}$ and the linear constraint sequence $\{c_i\}$ consists of purely real elements such as that in a real PAM transmission system. In that case (4.1.3) simplifies to

$$\sum_{i=-\infty}^{\infty} c_{-i}\theta_i = 1 \tag{4.1.4}$$

where $c_i, \; \theta_i \in \mathbb{R} \; \forall \; i$.

Also note that we adopt a negative index $-i$ without loss of generality for the constraint sequence elements $\{c_i\}$ in the linear parameter constraint given by (4.1.2). This will facilitate the expression of the constraint in an equivalent convolution form used in Chapter 5.

We also assume that the constraint sequence $\{c_i\}$ has *finite support*, *i.e.*,

$$c_i = 0, \quad \forall \; |i| > N \tag{4.1.5}$$

where N represents the degree of finite parametrisation given by (4.1.1). This measure is taken to ensure that the constraint is still met when the number of parameters is made finite using the practical equaliser implementation given by (4.1.1).

As demonstrated in Chapter 3, Section 3.3.2, the imposition of a suitable constraint on the equaliser parameters establishes a potential mechanism whereby the cost function can be minimised by a *particular* zero-ISI setting. In order that such a desirable property is realised, it is sufficient for the cost and constraint to align to the following ideals:

1. The *constraint set*, that is, the set of all equaliser parameter settings that satisfy (4.1.2)
$$\Theta \triangleq \left\{ \{\theta_i\} \middle| \sum_{i=-\infty}^{\infty} \mathrm{Re}\left\{\bar{c}_{-i}\theta_i\right\} = 1 \right\} \tag{4.1.6}$$
must contain at least one zero-ISI equaliser parameter setting.

2. If the constraint set contains more than one zero-ISI equaliser parameter setting, then those zero-ISI settings must produce corresponding equaliser outputs of the form
$$z_k = \gamma_\Delta a_{k-\Delta}, \quad \forall\, k \tag{4.1.7}$$
where the gain $\gamma_\Delta \in \mathbb{C}$ depends on the delay $\Delta \in \mathbb{Z}$.

   The zero-ISI equaliser settings, therefore, have the form
$$\theta_i^{*(\Delta)} = \gamma_\Delta \widetilde{h}_{i-\Delta} \quad \forall\, i \tag{4.1.8}$$
where $\{\widetilde{h}_i\}$ is the complex channel inverse impulse response.

   Our choice of a *linear* constraint (4.1.2) induces the following construction of the delay-dependent gain, γ_Δ. Substituting the general form of a zero-ISI equaliser parameter setting within the constraint set (4.1.8) into the parameter constraint equation in (4.1.2) we obtain
$$\mathrm{Re}\left\{\gamma_\Delta\right\} \sum_{i=-\infty}^{\infty} \mathrm{Re}\left\{\bar{c}_i \widetilde{h}_{i-\Delta}\right\} - \mathrm{Im}\left\{\gamma_\Delta\right\} \sum_{i=-\infty}^{\infty} \mathrm{Im}\left\{\bar{c}_i \widetilde{h}_{i-\Delta}\right\} = 1. \tag{4.1.9}$$
Note that the summations in (4.1.9) depends on the linear constraint sequence $\{c_i\}$ and the delayed channel inverse sequence $\{\widetilde{h}_{i-\Delta}\}$.

   For the real case, we substitute (4.1.8) into the real linear parameter constraint equation in (4.1.4) yielding
$$\gamma_\Delta = \frac{1}{\sum_{i=-\infty}^{\infty} c_i \widetilde{h}_{i-\Delta}}. \tag{4.1.10}$$

3. *Ideally*, the dependence of the gain γ_Δ on the delay Δ should be such that there exists a *unique* value of Δ for which $\mathcal{D}(\gamma_\Delta)$ is a minimum, that is,

$$\Delta^* \triangleq \arg\min_p \mathcal{D}(\gamma_p) \quad \text{is unique} \tag{4.1.11}$$

where $\mathcal{D}(\cdot)$ is some appropriate measure of the magnitude of the gain γ_Δ.

4. The cost function should be a monotonic increasing function of $\mathcal{D}(\gamma_\Delta)$. Combining this property with the ideal dependence of the gain γ_Δ on the delay Δ implies that such an average cost function $\mathcal{J}(\cdot)$ can be minimised *uniquely* by a *particular* zero-ISI equaliser parameter setting, that is,

$$\theta^{*(\Delta^*)} = \arg\min_{\theta\in\Theta} \mathcal{J}(\theta) \quad \text{is unique.} \tag{4.1.12}$$

5. For QAM systems, γ_Δ may, in addition to its magnitude, possess an unknown phase offset that represents a rotation of the open-eye diagram at the equaliser output relative to the original phase of the transmitted symbols.

In the design and analysis of a suitable cost and linear constraint, it will be convenient to pose the linearly-constrained cost minimisation problem in the *combined channel-equaliser parameter space* or *t-space* (*total parameter space*). In the remaining sections of this chapter we will make use of the following t-space relations that we can recall from Chapter 2, Section 2.3. The t-space parameters are defined by the impulse response $\{t_i\}$ of the cascade of the channel $\{h_i\}$ and equaliser $\{\theta_i\}$, namely,

$$t_k = h_k * \theta_k = \sum_{i=-\infty}^{\infty} \theta_i h_{k-i}, \quad \forall\, k \tag{4.1.13}$$

where $a * b$ denotes the convolution of the sequences a and b, and $a_k * b_k$ denotes the kth term in that convolution. The equaliser output $\{z_k\}$ can be expressed in terms of the t-space parameters as follows:

$$z_k = \sum_{i=-\infty}^{\infty} t_i a_{k-i}, \quad \forall\, k. \tag{4.1.14}$$

A mapping from the t-space parameters back to the equaliser parameters $\{\theta_i\}$ is given by

$$\theta_k = t_k * \widetilde{h}_k = \sum_{i=-\infty}^{\infty} t_i \widetilde{h}_{k-i}, \quad \forall\, k \tag{4.1.15}$$

where $\{\widetilde{h}_i\}$ denotes the impulse response of the inverse of the channel $\{h_i\}$.

4.2 Maximum Deviation Cost with Single Parameter Constraint

In this section, we reexamine a particular convex cost and simple linear constraint combination that we have already introduced in Chapter 3, Section 3.4. It was originally proposed in [29, 24, 26] and intended for blind equalisation in *real* PAM systems.

$$\Psi(z_k) = \max_{\{a_k\}} |z_k|, \quad \text{subject to } \theta_0 = 1. \tag{4.2.1}$$

In those works, it is shown that the combination of (4.2.1) is *admissible* in the generic case (subject to an inherent *nonuniqueness phenomenon* for specific degenerate channels). However, only minor intuitive justification was given for the particular choice of cost. For real PAM systems, we refer to the cost in (4.2.1) as a *maximum deviation cost.*

Here, we show that the maximum deviation cost is compatible with our previously stated requirement that the cost be a monotonic increasing function of the magnitude of the gain γ_Δ. We also provide a formal proof of the admissibility of the maximum deviation cost and single-parameter constraint combination. The proof uses a view of the constrained cost minimisation in the t-space rather than the equaliser parameter space. Finally, we show that, our first *naive* generalisation of the constrained maximum deviation cost to the complex QAM case is *inadmissible.*

4.2.1 Application to Real PAM Systems

In the developments to follow we assume that we are dealing with a real PAM system with a symmetric alphabet of the form

$$\mathcal{A} = \{-M,\ -M+2,\ \ldots,\ -3,\ -1,\ 1,\ 3,\ \ldots,\ M-2,\ M\} \tag{4.2.2}$$

One intuitively reasonable *measure* of the magnitude of the gain $\gamma_\Delta \in \mathbb{R}$ for real PAM systems is simply the *absolute value function*, formally,

$$\mathcal{D}(\cdot) \triangleq |\cdot|. \tag{4.2.3}$$

The cost must be a monotonically increasing function of $\mathcal{D}(\gamma_\Delta)$ and, at the same time, explicitly depend on observations of the equaliser output z_k. Since the EWGI criterion in (4.1.7) relates the zero-ISI equaliser output to the gain γ_Δ, we use this as a starting point to devise an appropriate cost function. Rearranging (4.1.7) and taking the *measure* of both sides gives us the following relation:

$$\mathcal{D}(\gamma_\Delta) = |\gamma_\Delta| = \frac{|z_k|}{|a_{k-\Delta}|}, \quad \forall\, k \tag{4.2.4}$$

At first glance, it appears impossible for any cost based solely on observations of the equaliser output, z_k, to reflect, in a monotonically increasing sense, the measure of the gain because the gain is a ratio of the equaliser output at time k and a symbol delayed by an *unknown* amount Δ relative to the equaliser output. To solve this problem, we take the tact of devising a cost that takes on a *particular* value of $|z_k|$ for which $\{a_k\}$ and hence $|a_{k-\Delta}|$ can be deduced. We now demonstrate that the convex cost of (4.2.1) fulfills this requirement. Using (4.1.14), which expresses the equaliser output in terms of the t-space parameters $\{t_i\}$ and the input symbol sequence $\{a_k\}$, we have

$$\max_{\{a_k\}} |z_k| = \max_{\{a_k\}} \left| \sum_{i=-\infty}^{\infty} t_i a_{k-i} \right|. \tag{4.2.5}$$

By Assumption 2.1 in Chapter 2,

$$\begin{aligned} \max_{\{a_k\}} |z_k| &= \left| \sum_{i=-\infty}^{\infty} t_i M \operatorname{sgn}(t_i) \right| \\ &= M \sum_{i=-\infty}^{\infty} |t_i| \end{aligned} \tag{4.2.6}$$

where $M = \max_{a \in \mathcal{A}} |a|$ of (4.2.2). That is, the maximum deviation of z_k corresponds to the case where $|a_k| = M \ \forall\, k$. Therefore we can construct the measure of the gain (4.2.4) as follows:

$$\mathcal{D}(\gamma_\Delta) = \frac{\max_{\{a_k\}} |z_k|}{M}. \tag{4.2.7}$$

Now, by simply rearranging (4.2.7), we obtain the proposed convex cost function.

$$\max_{\{a_k\}} |z_k| = M\mathcal{D}(\gamma_\Delta). \tag{4.2.8}$$

Note that the proposed convex cost function is indeed a monotonically increasing function of the measure of the gain, $\mathcal{D}(\gamma_\Delta)$ (in this case linear in $\mathcal{D}(\gamma_\Delta)$).

By our previous construction we have intuitively justified the maximum deviation cost of [29, 24, 26]. Furthermore, in those works, the admissibility of the maximum deviation cost and single parameter constraint combination (4.2.1) is established resulting in the following theorem:

Theorem 4.1 (Admissibility) *A double-sided infinite-length non-causal linear transversal equaliser with parameters $\{\theta_i\}$ operating on a channel with impulse response $\{h_i\}$ and channel inverse response $\{\tilde{h}_i\}$, driven by real data $\{a_k\}$ of the PAM type, and with parameter setting*

$$\theta_i = \frac{\tilde{h}_{i+m}}{\tilde{h}_m}, \tag{4.2.9}$$

where

$$m = \arg\max_p |\tilde{h}_p| \tag{4.2.10}$$

subject to the constraint $\theta_0 = 1$, minimises the blind convex cost

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k|.$$

We remark about Theorem 4.1 that the cost is minimised by a zero-ISI equaliser parameter setting of the form given by (4.1.8). To show this, we note that in order for the zero-ISI parameter setting to meet the constraint, γ_Δ is determined according to (4.1.10). Substituting (4.1.10) into (4.1.8) yields

$$\theta_i^{*(\Delta)} = \frac{\tilde{h}_{i-\Delta}}{\sum_{i=-\infty}^{\infty} c_{-i}\tilde{h}_{i-\Delta}} \quad \forall\, i \tag{4.2.11}$$

The constraint $\theta_0 = 1$ corresponds to the constraint sequence $\{c_i\} = \{\delta_i\}$, *i.e.*, the Kronecker delta sequence, in the general linear constraint equation (4.1.4). Substituting this constraint sequence into (4.2.11) and simplifying, we obtain

$$\theta_i^{*(\Delta)} = \frac{\tilde{h}_{i-\Delta}}{\tilde{h}_{-\Delta}}, \quad \forall\, i. \tag{4.2.12}$$

Therefore, we have arrived at the form of a zero-ISI equaliser parameter setting given by Theorem 4.1 where Δ corresponds to $-m$ of that theorem.

In the next section, we provide a formal proof of Theorem 4.1 using a more general method of proof than that of [29, 24, 26].

4.2.2 Constrained Minimisation in the Total Parameter Space

The problem relevant to Theorem 4.1 is one of minimising a cost function subject to a linear constraint in the equaliser parameter space. However, the form of the cost does not lend itself easily to expression as an *explicit* function of the equaliser parameters

$\{\theta_i\}$ and, therefore, it is difficult to characterise the minimum in the equaliser parameter space using general optimisation theory.

Here, we circumvent this difficulty by viewing the linearly-constrained cost minimisation problem in the t-space. The advantage gained is the fact that *both* the cost function and constraint can be expressed *explicitly* in terms of the t-space parameters and, therefore, the problem can be stated as a conventional optimisation problem in the t-space. Particular $\{t_i\}$ of interest can always be translated back to their representation in the equaliser parameter space by applying (4.1.15).

We have already seen in Section 4.2.1 that the *cost* of (4.2.1) can be expressed as function of the t-space parameters (4.2.6).

$$\mathcal{J}(\theta) = \mathcal{J}(t) = \max_{\{a_k\}} |z_k| = M \sum_{i=-\infty}^{\infty} |t_i|.$$

Since there is a one-to-one correspondence between the equaliser parameters $\{\theta_i\}$ and the corresponding t-space parameters $\{t_i\}$ (given by (4.1.13) and (4.1.15)), we can use the forms $\mathcal{J}(\theta)$ and $\mathcal{J}(t)$ interchangeably. As can be seen, the value of the cost for a particular equaliser parameter setting $\{\theta_i\}$, hence a particular $\{t_i\}$, is simply proportional to the l_1 *norm of* $\{t_i\}$. It should be noted that obtaining this form assumes that we are using a real PAM alphabet with maximum symbols $\pm M$ and that all finite subsequences of the symbol sequence are probable (Assumption 2.1 of Chapter 2).

The *constraint* of (4.2.1) can be expressed in the t-space using (4.1.15). Since

$$\theta_k = \sum_{i=-\infty}^{\infty} \widetilde{h}_{k-i} t_i$$

then,

$$\theta_0 = \sum_{i=-\infty}^{\infty} \widetilde{h}_{-i} t_i = 1. \tag{4.2.13}$$

Next, we prove Theorem 4.1 by using our new view of the problem. The following lemma is an intermediate result on the l_1 minimisation of an infinite-length sequence $\{t_i\}$ subject to a linear constraint on the sequence elements. The proof of Theorem 4.1 will subsequently make use of this lemma.

Lemma 4.1 *A solution to the problem,*

$$\underset{\{t_i\}}{\text{minimize}} \quad \sum_{i=-\infty}^{\infty} |t_i|$$

$$\text{subject to} \quad \sum_{i=-\infty}^{\infty} c_i t_i = 1$$

where $t_i,\ c_i \in \mathbb{R}$, *has the form*

$$t_i^* = \begin{cases} 1/c_r, & i = r \triangleq \arg\max_p |c_p| \\ 0, & i \neq r. \end{cases}$$

Proof: Define

$$\mathcal{J}(t) \triangleq \sum_{i=-\infty}^{\infty} |t_i|, \quad t \triangleq \{t_i\} \tag{4.2.14}$$

and let t^* denote the candidate minimising sequence as follows:

$$t^* = \left\{ \ \dots \ 0 \ \ t_r^* \ \ 0 \ \dots \right\}. \tag{4.2.15}$$

Then, to satisfy the constraint,

$$t_r^* = \frac{1}{c_r}. \tag{4.2.16}$$

Therefore,

$$\mathcal{J}(t^*) = \frac{1}{|c_r|}. \tag{4.2.17}$$

Let the sequence, Δt, represent a perturbation of the candidate sequence such that $t^* + \Delta t$ satisfies the constraint, *i.e.*,

$$\sum_{i=-\infty}^{\infty} c_i (t_i^* + \Delta t_i) = 1, \tag{4.2.18}$$

then it follows that

$$\sum_{i=-\infty}^{\infty} c_i \Delta t_i = 0. \tag{4.2.19}$$

The incremental cost is given by

$$\Delta\mathcal{J} \triangleq \mathcal{J}(t^* + \Delta t) - \mathcal{J}(t^*) = \sum_{i \neq r} |\Delta t_i| + \left| \frac{1}{c_r} + \Delta t_r \right| - \left| \frac{1}{c_r} \right|. \tag{4.2.20}$$

Multiplying both sides of (4.2.20) by $|c_j|$ and simplifying gives

$$|c_r|\Delta\mathcal{J} \quad = \quad \sum_{i \neq r} |c_r \Delta t_i| + |1 + \Delta t_r c_r| - 1$$

$$
\begin{aligned}
&\geq \sum_{i \neq r} c_i \Delta t_i + \Delta t_r c_r \\
&= \sum_{i=-\infty}^{\infty} c_i \Delta t_i = 0
\end{aligned}
\tag{4.2.21}
$$

where we have used (4.2.19) and the following facts:

By our definition of r, $|c_r| \geq |c_i| \; \forall i$, therefore

$$
\sum_{i \neq r} |c_r \Delta t_i| \geq \sum_{i \neq r} c_i \Delta t_i. \tag{4.2.22}
$$

Also,

$$
|1+\epsilon| - 1 \geq \epsilon, \quad \epsilon \in \mathbb{R}. \tag{4.2.23}
$$

∎

We are now in a position to give a formal proof of Theorem 4.1.

Proof of Theorem 4.1

Proof: We begin with the form of cost in the t-space given by (4.2.6)

$$
\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k| = M \sum_{i=-\infty}^{\infty} |t_i|
$$

where $M = \max_{a \in \mathcal{A}} |a|$.

The constraint, $\theta_0 = 1$, can be also expressed in the t-space using (4.1.15), yielding

$$
\theta_0 = \sum_{i=-\infty}^{\infty} \widetilde{h}_{-i} t_i = 1. \tag{4.2.24}
$$

The cost and the constraint are now in a form where Lemma 4.1 can be applied. In this case,

$$
t_i^* = \begin{cases} 1/\widetilde{h}_m, & i = -m, \quad m = \arg\max_p |\widetilde{h}_p| \\ 0, & i \neq -m \end{cases} \tag{4.2.25}
$$

or equivalently,

$$
t_i^* = h_i * \theta_i^* = \frac{1}{\widetilde{h}_m} \delta(i+m) \tag{4.2.26}
$$

which implies that

$$
\theta_i^* = \widetilde{h}_i * \frac{1}{\widetilde{h}_m} \delta(i+m) = \frac{\widetilde{h}_{i+m}}{\widetilde{h}_m}. \tag{4.2.27}
$$

∎

4.2.3 Application to Complex QAM Systems

As a first *naive* generalisation of the admissible cost and constraint of (4.2.1) to the case of complex QAM alphabets, we simply interpret the "$|\cdot|$" operation to be the *complex modulus* instead of the absolute value. Without any analysis, this generalisation naturally appears to be the complex analogue of (4.2.1). Since minimisation of this *maximum modulus* cost corresponds to forcing the eye-diagram to have a smaller uniform modulus, it seems suited for *phase shift keying* (PSK) constellations. However, the following result establishes the inadmissibility of this simple generalisation.

Theorem 4.2 Inadmissibility *A double-sided non-causal linear transversal equaliser with complex parameters $\{\theta_i\}$ operating on a complex channel with impulse response $\{h_i\}$ and channel inverse response $\{\tilde{h}_i\}$, driven by symbols $\{a_k\}$ from a PSK-type symbol constellation, and with a parameter setting given by*

$$\frac{\tilde{h}_{i+m}}{\tilde{h}_m}, \tag{4.2.28}$$

where $\{\tilde{h}_i\}$ is the inverse of $\{h_i\}$ and

$$m = \arg\max_p |\tilde{h}_p| \tag{4.2.29}$$

fails to minimise the maximum modulus cost

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k|$$

subject to the constraint $\theta_0 = 1$ and is therefore inadmissible.

Proof: Consider the case where the channel inverse, $\{\tilde{h}_i\}$ is such that the delay m in (4.2.29) is *not* unique. Specifically, suppose that there are two delays m_1 and m_2 such that

$$|\tilde{h}_{m_1}| = |\tilde{h}_{m_2}| = \max_p |\tilde{h}_p| = g. \tag{4.2.30}$$

In other words, the channel inverse $\{h_i\}$ has two identical *peaks* in magnitude.[1] Then we have two zero-ISI equaliser parameter settings in the constraint set for $\theta_0 = 1$ that

[1] or Twin Peaks.

potentially minimise the cost, namely,

$$\theta_i^{(1)} = \frac{\widetilde{h}_{i+m_1}}{\widetilde{h}_{m_1}} = \frac{\widetilde{h}_{i+m_1}}{g} e^{j\phi_{m_1}} \quad \forall\ i \tag{4.2.31}$$

$$\theta_i^{(2)} = \frac{\widetilde{h}_{i+m_2}}{g} e^{j\phi_{m_2}} \quad \forall\ i \tag{4.2.32}$$

where $\phi_{m_j} = -\arg(\widetilde{h}_{m_j})$, $j \in \{1,2\}$. The corresponding equaliser outputs are

$$\begin{aligned} z_k^{(1)} &= h_k * \theta_k^{(1)} * a_k = h_k * \frac{\widetilde{h}_{k+m_1}}{g} e^{j\phi_{m_1}} * a_k \\ &= \frac{\delta_{k+m_1}}{g} e^{j\phi_{m_1}} * a_k \\ &= \frac{a_{k+m_1}}{g} e^{j\phi_{m_1}} \end{aligned} \tag{4.2.33}$$

$$z_k^{(2)} = \frac{a_{k+m_2}}{g} e^{j\phi_{m_2}} \tag{4.2.34}$$

and their associated cost function values are

$$\mathcal{J}(\theta^{(1)}) = \max_{\{a_k\}} |z_k^{(1)}| = \frac{M}{g} \tag{4.2.35}$$

$$\mathcal{J}(\theta^{(2)}) = \frac{M}{g} \tag{4.2.36}$$

where M is the constant modulus of the PSK symbols. Note that the two candidate cost-minimising zero-ISI equaliser parameter settings of (4.2.31)-(4.2.32) correspond to identical values of the cost. We now consider a *convex* combination of these two candidate settings that is effectively between the two settings in the parameter space, *viz.*,

$$\theta_i(\lambda) = (1-\lambda)\theta_i^{(1)} + \lambda\theta_i^{(2)}, \quad \forall\ i \tag{4.2.37}$$

where $0 < \lambda < 1$. Note that this convex combination of parameter settings will always remain in the constraint set as evidenced by the following:

$$\begin{aligned} \theta_0(\lambda) &= (1-\lambda)\theta_0^{(1)} + \lambda\theta_0^{(2)} \\ &= (1-\lambda)(1) + \lambda(1) \\ &= 1. \end{aligned} \tag{4.2.38}$$

The equaliser output corresponding to $\theta(\lambda)$ is

$$\begin{aligned} z_k &= (1-\lambda)\frac{a_{k+m_1}}{g} e^{j\phi_{m_1}} + \lambda\frac{a_{k+m_2}}{g} e^{j\phi_{m_2}} \\ &= (1-\lambda)\frac{M}{g} e^{j\phi'_{m_1}(k)} + \lambda\frac{M}{g} e^{j\phi'_{m_2}(k)} \\ &= \frac{M}{g}\left[(1-\lambda) + \lambda e^{j\phi'(k)}\right] e^{j\phi'_{m_1}(k)} \end{aligned} \tag{4.2.39}$$

where

$$\phi'_{m_j}(k) = \phi_{m_j} + \arg(a_{k+m_j}), \quad j \in \{1,2\} \tag{4.2.40}$$

and $\phi'(k) = \phi'_{m_2}(k) - \phi'_{m_1}(k)$. By construction, we also require that

$$|\phi'(k)| < \pi, \quad \forall\, k. \tag{4.2.41}$$

Then the cost value between these two settings is given by

$$\mathcal{J}(\theta(\lambda)) = \max_{\{a_k\}} |z_k| = \frac{M}{g} \left|(1-\lambda) + \lambda e^{j\phi'}\right| \tag{4.2.42}$$

where $\phi' = \phi'(k)$ of the particular maximising symbol sequence $\{a_k\}$. Using the law of cosines to evaluate the modulus in (4.2.42) we obtain

$$\mathcal{J}(\theta(\lambda)) = \frac{M}{g} \left[(1-\lambda)^2 + \lambda^2 - 2\lambda(1-\lambda)\cos(\phi')\right]^{\frac{1}{2}} \tag{4.2.43}$$

which has a unique minimum value at $\lambda = 0.5$, namely,

$$\mathcal{J}(\theta(0.5)) = \frac{M}{g}\cos\left(\frac{\phi'}{2}\right). \tag{4.2.44}$$

However, since $|\phi'| < \pi$ we have that

$$\mathcal{J}(\theta(0.5)) < \frac{M}{g} = \mathcal{J}(\theta^{(1)}) = \mathcal{J}(\theta^{(2)}). \tag{4.2.45}$$

The equaliser output corresponding to the parametrisation, $\theta(0.5)$ may be expressed as follows:

$$z_k = \frac{1}{2g}\left[a_{k+m_1} e^{j\phi_{m_1}} + a_{k+m_2} e^{j\phi_{m_2}}\right]. \tag{4.2.46}$$

Note that the form of (4.2.46) technically corresponds to that of a closed-eye parameter setting. Therefore, we have demonstrated inadmissibility in this case by showing that a closed-eye equaliser parameter setting achieves a lower cost than the zero-ISI settings. ∎

Remarks

1. Using a construction similar to that of the proof, the inadmissibility result can be proven for the case where the channel inverse sequence possesses an *arbitrary* number of identical peaks in the sense of (4.2.30). For example, if the channel inverse possesses three identical peaks then the resulting equaliser output will be

of the form (4.2.46) but may consist of the sum of three terms corresponding to three adjacent symbols. Hence the corresponding eye diagram will be even more closed than that resulting from two identical peaks.

2. The proof of Theorem 4.2 assumes a rather rare condition of the channel inverse sequence possessing twin peaks (4.2.30). However, the result is still true for the more probable case in which the peaks are simply *close* in magnitude.

   Instead of (4.2.30), suppose we have

$$|\tilde{h}_{m_1}| = \max_p |\tilde{h}_p| = g \tag{4.2.47}$$

$$|\tilde{h}_{m_2}| = (1-\epsilon)g \tag{4.2.48}$$

   where $\epsilon \in \mathbb{R}$ is small. Forming the two corresponding zero-ISI equaliser parameter settings in the constraint set we obtain

$$\theta_i^{(1)} = \frac{\tilde{h}_{i+m_1}}{g} e^{j\phi_{m_1}} \quad \forall\, i \tag{4.2.49}$$

$$\theta_i^{(2)} = \frac{\tilde{h}_{i+m_2}}{(1-\epsilon)g} e^{j\phi_{m_2}} \quad \forall\, i. \tag{4.2.50}$$

   Forming the convex combination $\theta(\lambda)$ of $\theta^{(1)}$ and $\theta^{(2)}$ and evaluating the corresponding equaliser output we obtain

$$z_k = \frac{M}{g}\left[(1-\lambda) + \frac{\lambda}{(1-\epsilon)} e^{j\phi'(k)}\right] e^{j\phi'_{m_1}(k)}. \tag{4.2.51}$$

   The cost for this equaliser parametrisation is

$$\mathcal{J}(\theta(\lambda)) = \frac{M}{g}\left|(1-\lambda) + \frac{\lambda}{(1-\epsilon)} e^{j\phi'}\right| < J(\theta^{(1)}). \tag{4.2.52}$$

   The cost $J(\theta^{(1)})$ has a unique minimum for some $0 < \lambda < 0.5$, specifically,

$$\lambda = \frac{1 - \frac{1}{(1-\epsilon)}\cos(\phi')}{1 + \frac{1}{(1-\epsilon)^2} - \frac{2}{(1-\epsilon)}\cos(\phi')}. \tag{4.2.53}$$

We may illustrate the inadmissibility result by simulating a problem channel that is constructed so that its inverse has two peaks that are close in magnitude. The complex impulse response of the channel is shown in Figure 4-1 and the magnitude of its inverse impulse response is shown in Figure 4-2.

A simulation of our blind equaliser algorithm adopting the maximum modulus cost and single parameter constraint $\theta_0 = 1$ was run for 4-PSK (QAM) symbols transmitted through the channel in Figure 4-1.

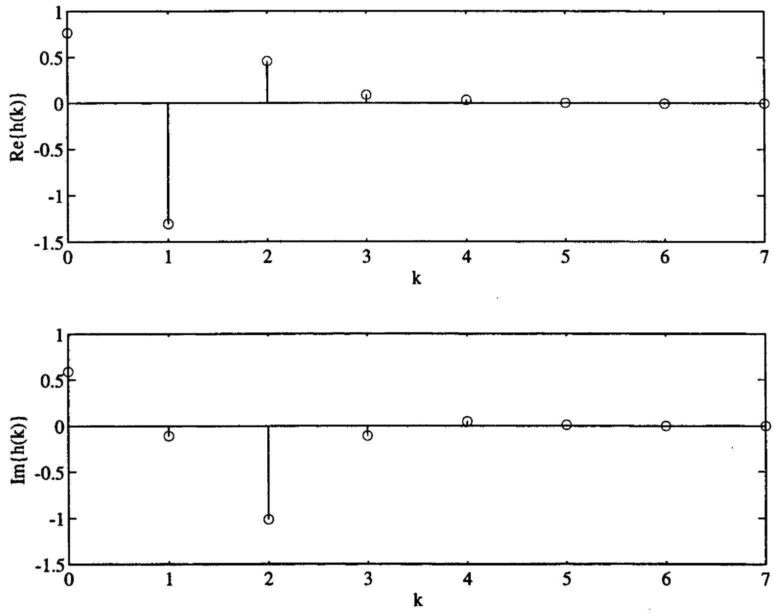


Figure 4-1: The impulse response of the problem channel.

Figure 4-3 gives the eye diagrams of the channel and equaliser output after the cost is minimised. It can be seen that the equaliser parameters have converged to a closed-eye equaliser parameter setting. The evolution of the intersymbol interference and the maximum modulus cost over the iterations of the algorithm is shown in Figure 4-4.

As remarked, Theorem 4.2 establishes the inadmissibility of the maximum modulus cost and single-parameter constraint for a class of channels whose channel inverse sequence possesses two or more peaks in magnitude that are approximately the same. We now investigate the case where we do not have such channels, *i.e.*, the channel inverse sequence possesses a clearly dominant peak in magnitude and hence there exists a *unique* m in (4.2.10). Using a method of proof similar to that of Theorem 4.1, it can be shown that in that case, the maximum modulus cost *can* be minimised by an equaliser parameter setting given by (4.2.28) and (4.2.29). However, if we retrace the first few steps of the previous proof assuming that we have a unique delay m, then we obtain the following expression of the equaliser output

$$z_k = \frac{a_{k+m}}{\widetilde{h}_m} = \frac{a_{k+m}}{|\widetilde{h}_m|} e^{j\phi_m} \tag{4.2.54}$$

where $\phi_m = -\arg(\widetilde{h}_m)$. It is clear from (4.2.54) that the equaliser output is free of ISI, however, it has a phase offset of ϕ_m due to the phase of the $\widetilde{h}_m$. The eye-diagram of the

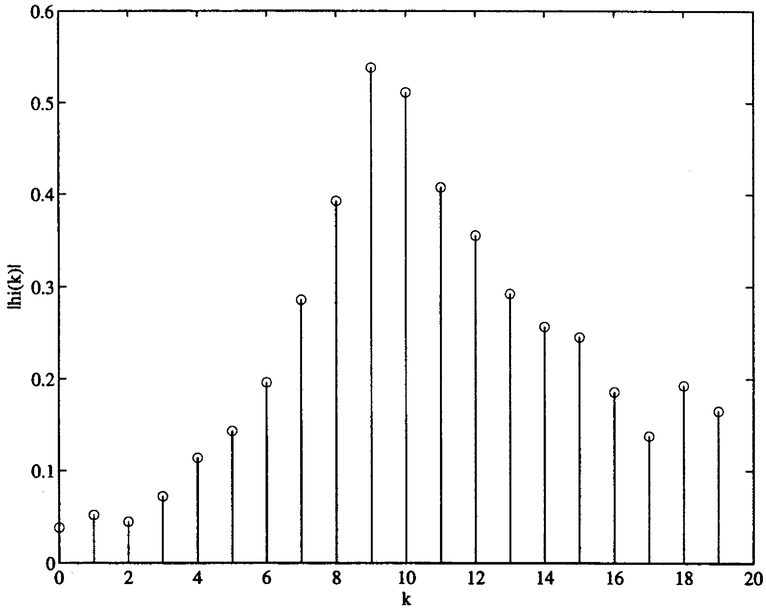


Figure 4-2: The magnitude of the impulse response of the problem channel inverse. Note that $|\widetilde{h}(10)| = |(1-\epsilon)\widetilde{h}(9)|$ where $\epsilon = 0.05$.

equaliser output in this case will be rotated by this amount. Because of the inability of the equaliser to account for the phase offset, this makes necessary the use of a separate phase recovery stage following the equaliser and prior to the quantiser.

It is important to note that this phase offset is a result of the normalisation of the delayed channel inverse sequence by $\widetilde{h}_m$ so that the constraint of $\theta_0 = 1$ is met. Therefore, we see that it is the rigidity of the phase of θ_0 enforced by the constraint that is preventing the equaliser from being able to recover the phase offset. It may therefore be desirable to use a different constraint for complex QAM systems that allows the equaliser parameters to take on an arbitrary phase.

Figure 4-5 gives the eye-diagrams resulting from our 4-PSK simulation when the channel inverse has a dominant peak in magnitude. The complex channel (4.2.55) is a first-order real allpass channel with an added phase offset of $\pi/4$.

$$H(z^{-1}) = \frac{0.7 - z^{-1}}{1 - 0.7z^{-1}} e^{j\frac{\pi}{4}} \tag{4.2.55}$$

The corresponding ISI and cost evolutions are shown in Figure 4-6.

4.3 Maximum Complex Deviation Cost with Single Parameter Constraint

In this section, we motivate the development of a cost and single-parameter constraint that takes into consideration the *phase* of the unknown gain factor γ (4.1.7): a factor not considered in our naive generalisation of Section 4.2. We have found that fixing the phase of the constraint as in $\theta_0 = 1$ prevents the equaliser from being able to recover the phase of γ.

As discussed in Section 4.1, we need to adopt an appropriate measure $\mathcal{D}(\cdot)$ for the gain, γ_Δ. For real systems, the measure $\mathcal{D}(\cdot) = |\cdot|$ (4.2.3) is appropriate. However, it may be considered unsatisfactory for complex systems because it does not penalise the possible deviations in phase of γ_Δ away from acceptable phase offsets which represent *isomorphic* rotations of the transmitted symbol constellation. For example, for most QAM constellations such as 16-QAM and 64-QAM, any multiple of $\pi/2$ is an acceptable offset.

To simplify the following development, we will assume that the QAM constellation is *square*. This implies that the constellation has symmetrical "*corner*" points. For example, 16-QAM and 64-QAM constellations are square whilst 32-QAM is not.

Definition 4.1 (Square Constellation) *A square symbol constellation $\mathcal{A}$ is one where*

$$\pm M \pm Mj \in \mathcal{A}, \quad M = \max_{a \in \mathcal{A}} |Re\{a\}| = \max_{a \in \mathcal{A}} |Im\{a\}|. \tag{4.3.1}$$

A suitable measure of the gain in this context is

$$\mathcal{D}(\gamma_\Delta) = |\mathrm{Re}\{\gamma_\Delta\}| + |\mathrm{Im}\{\gamma_\Delta\}|. \tag{4.3.2}$$

Figure 4-7 gives a plot of $\mathcal{D}(\gamma_\Delta)$ as a function of $\arg(\gamma_\Delta)$ where its magnitude $|\gamma_\Delta|$ is assumed to be unity.

The measure in (4.3.2) has minimum and maximum values as follows:

$$\mathcal{D}(\gamma_\Delta) = \begin{cases} |\gamma_\Delta|, & \arg(\gamma_\Delta) = \frac{k\pi}{2} \\ \sqrt{2}|\gamma_\Delta|, & \arg(\gamma_\Delta) = \frac{\pi}{4} + \frac{k\pi}{2} \end{cases} \tag{4.3.3}$$

for $k \in \{0, 1, 2, 3\}$. Therefore, it correctly penalises the deviations of $\arg(\gamma_\Delta)$ away from the isomorphic rotations. We may follow the same procedure as in Section 4.2.1

to devise an appropriate cost function. Rearranging (4.1.7) as before and taking the new measure of both sides yields

$$
\begin{aligned}
\mathcal{D}(\gamma_\Delta) &= \left|\mathrm{Re}\left\{\frac{z_k}{a_{k-\Delta}}\right\}\right| + \left|\mathrm{Im}\left\{\frac{z_k}{a_{k-\Delta}}\right\}\right| \\
&= \left|\mathrm{Re}\{z_k\}\,\mathrm{Re}\left\{\frac{\bar{a}_{k-\Delta}}{|a_{k-\Delta}|^2}\right\} - \mathrm{Im}\{z_k\}\,\mathrm{Im}\left\{\frac{\bar{a}_{k-\Delta}}{|a_{k-\Delta}|^2}\right\}\right| \\
&\quad + \left|\mathrm{Re}\{z_k\}\,\mathrm{Im}\left\{\frac{\bar{a}_{k-\Delta}}{|a_{k-\Delta}|^2}\right\} + \mathrm{Im}\{z_k\}\,\mathrm{Re}\left\{\frac{\bar{a}_{k-\Delta}}{|a_{k-\Delta}|^2}\right\}\right| \\
&= \frac{1}{|a_{k-\Delta}|^2}\left[|\mathrm{Re}\{z_k\}\,\mathrm{Re}\{a_{k-\Delta}\} + \mathrm{Im}\{z_k\}\,\mathrm{Im}\{a_{k-\Delta}\}|\right. \\
&\quad \left. + |-\mathrm{Re}\{z_k\}\,\mathrm{Im}\{a_{k-\Delta}\} + \mathrm{Im}\{z_k\}\,\mathrm{Re}\{a_{k-\Delta}\}|\right].
\end{aligned}
\tag{4.3.4}
$$

Now, we look at a *particular* form of the equaliser output z_k at time for which $|a_{k-\Delta}|$ is known. One quantity that we can observe for which $|a_{k-\Delta}|$ is known is

$$
\max_{\{a_k\}} |\mathrm{Re}\{z_k\}|. \tag{4.3.5}
$$

This becomes evident when we simplify (4.3.5) using the form of the equaliser output given in terms of the t-space parameters from (4.1.14) and using our *square* constellation assumption and Assumption 2.1 to perform the maximisation. Explicitly,

$$
\begin{aligned}
\max_{\{a_k\}} |\mathrm{Re}\{z_k\}| &= \max_{\{a_k\}} \left|\mathrm{Re}\left\{\sum_{i=-\infty}^{\infty} t_i a_{k-i}\right\}\right| \\
&= \max_{\{a_k\}} \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\}\,\mathrm{Re}\{a_{k-i}\} - \mathrm{Im}\{t_i\}\,\mathrm{Im}\{a_{k-i}\}\right| \\
&= \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\}\,M\mathrm{sgn}(\mathrm{Re}\{t_i\}) - \mathrm{Im}\{t_i\}\,(-M\mathrm{sgn}(\mathrm{Im}\{t_i\}))\right| \\
&= M \sum_{i=-\infty}^{\infty} |\mathrm{Re}\{t_i\}| + |\mathrm{Im}\{t_i\}|
\end{aligned}
\tag{4.3.6}
$$

where the maximising sequence $\{a_k^*\}$ is given by

$$
a_{k-i}^* = M\mathrm{sgn}(\mathrm{Re}\{t_i\}) - jM\mathrm{sgn}(\mathrm{Im}\{t_i\}), \quad \forall\, i. \tag{4.3.7}
$$

Substituting the maximising symbols of (4.3.7) into (4.3.4) yields

$$
\begin{aligned}
\mathcal{D}(\gamma_\Delta) &= \frac{1}{2M}\left[\left|\max_{\{a_k\}} |\mathrm{Re}\{z_k\}|\,\mathrm{sgn}(\mathrm{Re}\{t_i\}) - \mathrm{Im}\{z_k\}\,\mathrm{sgn}(\mathrm{Im}\{t_i\})\right|\right. \\
&\quad \left. + \left|\max_{\{a_k\}} |\mathrm{Re}\{z_k\}|\,\mathrm{sgn}(\mathrm{Im}\{t_i\}) + \mathrm{Im}\{z_k\}\,\mathrm{sgn}(\mathrm{Re}\{t_i\})\right|\right] \\
&= \frac{1}{2M}\left[\left|\max_{\{a_k\}} |\mathrm{Re}\{z_k\}| - \mathrm{Im}\{z_k\}\right| + \left|\max_{\{a_k\}} |\mathrm{Re}\{z_k\}| + \mathrm{Im}\{z_k\}\right|\right]
\end{aligned}
\tag{4.3.8}
$$

where the imaginary part of the equaliser output is given by

$$\begin{aligned}
\mathrm{Im}\{z_k\} &= \sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\}\,\mathrm{Im}\{a^*_{k-i}\} + \mathrm{Re}\{a^*_{k-i}\}\,\mathrm{Im}\{t_i\} \\
&= M \sum_{i=-\infty}^{\infty} -\mathrm{Re}\{t_i\}\,\mathrm{sgn}(\mathrm{Im}\{t_i\}) + \mathrm{Im}\{t_i\}\,\mathrm{sgn}(\mathrm{Re}\{t_i\}) \\
&\leq \max_{\{a_k\}} |\mathrm{Re}\{z_k\}|. \qquad (4.3.9)
\end{aligned}$$

Equality holds in (4.3.9) when

$$\mathrm{sgn}(\mathrm{Re}\{t_i\}) = -\mathrm{sgn}(\mathrm{Re}\{t_i\}), \quad \forall\, i, \tag{4.3.10}$$

or equivalently when

$$\mathrm{Re}\{t_i\} = 0, \quad \forall\, i. \tag{4.3.11}$$

Given the inequality of (4.3.9), the expression in (4.3.8) can be further simplified using the relation

$$|a+b| + |a-b| = 2\max(|a|, |b|), \quad a,\ b \in \mathbb{R}. \tag{4.3.12}$$

The expression in (4.3.8) simplifies to

$$\mathcal{D}(\gamma_\Delta) = \frac{\max_{\{a_k\}} |\mathrm{Re}\{z_k\}|}{M}. \tag{4.3.13}$$

Rearranging (4.3.13), we obtain the following cost function which we call the *maximum complex deviation cost.*

$$\max_{\{a_k\}} |\mathrm{Re}\{z_k\}| = M\mathcal{D}(\gamma_\Delta). \tag{4.3.14}$$

Note that the cost function is a monotonically increasing function of the measure of the gain $\mathcal{D}(\gamma_\Delta)$.

Remarks

1. The *maximum complex deviation cost* of (4.3.14) is, in fact, one that has been proposed in [34] and found *admissible* for square QAM transmission systems. We have therefore justified its use using our design philosophy.

2. An alternative form of the equaliser output to (4.3.5) for which $|a_{k-\Delta}|$ is also known is

$$\max_{\{a_k\}} |\mathrm{Im}\{z_k\}|.$$

Using the same method of derivation we can obtain an equivalent maximum complex deviation cost to (4.3.14), namely,

$$\max_{\{a_k\}} |\text{Im}\{z_k\}| = M\mathcal{D}(\gamma_\Delta). \tag{4.3.15}$$

We may therefore combine the costs of (4.3.14) and (4.3.15) to form

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |\text{Re}\{z_k\}| + \max_{\{a_k\}} |\text{Im}\{z_k\}| \tag{4.3.16}$$

where, in general, the maximising symbol sequences for the real and imaginary parts of the equaliser output are different. This cost combination is also presented in [34].

A suitable linear equaliser parameter constraint for the cost of (4.3.16) (and of [34]) was proposed in [34], namely,

$$\text{Re}\{\theta_0\} + \text{Im}\{\theta_0\} = 1. \tag{4.3.17}$$

Unlike the constraint $\theta_0 = 1$ the linear constraint of (4.3.17) allows the zeroth equaliser parameter to potentially possess a phase between $-\pi/4$ and $3\pi/4$. Therefore, we would expect that this would allow the equaliser parameters to remove an arbitrary phase offset of a square symbol constellation. The following theorem on the *admissibility* of the combination of the cost (4.3.16) and linear constraint (4.3.17) was established in [34].

Theorem 4.3 *A double-sided infinite-length non-causal linear transversal equaliser with parameters $\{\theta_i\}$ operating on a channel with impulse response $\{h_i\}$ and channel inverse response $\{\tilde{h}_i\}$ driven by complex data $\{a_k\}$ of the square type (or transformed to such via rotation), and with parameter setting*

$$\theta_i = \frac{\tilde{h}_{i+m} e^{jk\pi/2}}{\left|Re\left\{\tilde{h}_m\right\}\right| + \left|Im\left\{\tilde{h}_m\right\}\right|}, \quad k \in \{0, 1, 2, 3\},$$

where

$$m = \arg\max_p \left\{\left|Re\left\{\tilde{h}_p\right\}\right| + \left|Im\left\{\tilde{h}_p\right\}\right|\right\},$$

subject to the constraint $Re\{\theta_0\} + Im\{\theta_0\} = 1$, minimises the blind convex cost

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |Re\{z_k\}| + \max_{\{a_k\}} |Im\{z_k\}|.$$

The proof of Theorem 4.3 in [34] was complex and did not provide much general insight into the linearly-constrained cost minimisation problem.

4.3.1 Constrained Minimisation in the Total Parameter Space

In a similar manner to Section 4.2.2, we present a proof of Theorem 4.3 by expressing the constrained minimisation problem in in the t-space. The proof derives the following expressions of the cost and constraint in terms of t-space parameters:

$$\mathcal{J}(t) = 2M \sum_{i=-\infty}^{\infty} |\mathrm{Re}\{t_i\}| + |\mathrm{Im}\{t_i\}|$$

and

$$\sum_{i=-\infty}^{\infty} \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} + \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Re}\{t_i\} + \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} - \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Im}\{t_i\} = 1.$$

We remark that the value of the cost for a particular complex sequence $\{t_i\}$ is the l_1 norm of a corresponding real sequence that consists of the real and imaginary components of the complex sequence. The above expression of the cost is valid assuming that Assumption 2.1 holds as before and the symbol constellation is *square*. Noting the form of the cost and the constraint in the t-space, the following lemma gives the general solution to a constrained minimisation problem of that form. It is subsequently required by the proof of Theorem 4.3.

Lemma 4.2 *A solution to the problem,*

$$\begin{array}{ll} \underset{\{t_i\}}{\textit{minimize}} & \sum_{i=-\infty}^{\infty} |Re\{t_i\}| + |Im\{t_i\}| \\ \textit{subject to} & \sum_{i=-\infty}^{\infty} a_i Re\{t_i\} + b_i Im\{t_i\} = 1 \end{array}$$

where $t_i \in \mathbb{C}$ and $a_i, b_i \in \mathbb{R}$ has the form

$$t_i^* = \begin{cases} t_r^*, & i = r \\ 0, & i \neq r \end{cases}$$

where

$$t_r^* = \begin{cases} 1/a_r, & |a_r| > |b_r| \\ j/b_r, & |a_r| < |b_r| \end{cases}$$

and

$$r = \arg\max_p \{\max[|a_p|, |b_p|]\}.$$

Proof: Consider the sequence, $\{t'_i\}, t'_i \in \mathbb{R}$, defined as

$$t'_{2i} \triangleq \mathrm{Re}\{t_i\}, \quad t'_{2i+1} \triangleq \mathrm{Im}\{t_i\}, \quad \forall\, i$$

and the constraint sequence, $\{c_i\}, c_i \in \mathbb{R}$, defined as

$$c_{2i} \triangleq a_i, \quad c_{2i+1} \triangleq b_i, \quad \forall\, i.$$

Then, the problem can be restated as

$$\begin{array}{ll} \underset{\{t'_i\}}{\text{minimize}} & \sum_{i=-\infty}^{\infty} |t'_i| \\ \text{subject to} & \sum_{i=-\infty}^{\infty} c_i t'_i = 1 \end{array}$$

which is of the form of the real case (Lemma 4.1) and hence the result follows. ∎

Lemma 4.2 is now used in the following formal proof of Theorem 4.3.

Proof of Theorem 4.3

Proof: For square QAM symbol sequences obeying Assumption 2.1, the equivalent t-space representation of $\mathcal{J}(\theta)$ can be derived as follows:

$$\begin{aligned} \mathcal{J}(\theta) &= \max_{\{a_k\}} |\mathrm{Re}\{z_k\}| + \max_{\{a_k\}} |\mathrm{Im}\{z_k\}| \\ &= \max_{\{a_k\}} \left|\mathrm{Re}\left\{\sum_{i=-\infty}^{\infty} t_i a_{k-i}\right\}\right| + \max_{\{a_k\}} \left|\mathrm{Im}\left\{\sum_{i=-\infty}^{\infty} t_i a_{k-i}\right\}\right| \\ &= \max_{\{a_k\}} \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\}\mathrm{Re}\{a_{k-i}\} - \mathrm{Im}\{t_i\}\mathrm{Im}\{a_{k-i}\}\right| \\ &\quad + \max_{\{a_k\}} \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\}\mathrm{Im}\{a_{k-i}\} + \mathrm{Re}\{a_{k-i}\}\mathrm{Im}\{t_i\}\right| \\ &= \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\} M\mathrm{sgn}(\mathrm{Re}\{t_i\}) - \mathrm{Im}\{t_i\}(-M\mathrm{sgn}(\mathrm{Im}\{t_i\}))\right| \\ &\quad + \left|\sum_{i=-\infty}^{\infty} \mathrm{Re}\{t_i\} M\mathrm{sgn}(\mathrm{Re}\{t_i\}) + \mathrm{Im}\{t_i\} M\mathrm{sgn}(\mathrm{Im}\{t_i\})\right| \\ &= 2M \sum_{i=-\infty}^{\infty} |\mathrm{Re}\{t_i\}| + |\mathrm{Im}\{t_i\}|. \end{aligned} \tag{4.3.18}$$

The constraint, $\mathrm{Re}\{\theta_0\} + \mathrm{Im}\{\theta_0\} = 1$ can be also expressed in the t-space. Since

$$\mathrm{Re}\{\theta_k\} + \mathrm{Im}\{\theta_k\} = \sum_{i=-\infty}^{\infty} \mathrm{Re}\left\{\widetilde{h}_{k-i}t_i\right\} + \mathrm{Im}\left\{\widetilde{h}_{k-i}t_i\right\}$$

then

$$\mathrm{Re}\{\theta_0\} + \mathrm{Im}\{\theta_0\} = \sum_{i=-\infty}^{\infty} \mathrm{Re}\left\{\tilde{h}_{-i} t_i\right\} + \mathrm{Im}\left\{\tilde{h}_{-i} t_i\right\} = 1$$

or

$$\sum_{i=-\infty}^{\infty} \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} + \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Re}\{t_i\} + \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} - \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Im}\{t_i\} = 1. \quad (4.3.19)$$

The cost and the constraint are now in a form where Lemma 4.2 can be applied. In this case,

$$t_i^* = \begin{cases} t_{-m}^* & i = -m \\ 0, & i \neq -m \end{cases}$$

where

$$t_{-m}^* = \begin{cases} 1/\left(\mathrm{Re}\left\{\tilde{h}_m\right\} + \mathrm{Im}\left\{\tilde{h}_m\right\}\right), & \chi_m^{(+)} > \chi_m^{(-)} \\ j/\left(\mathrm{Re}\left\{\tilde{h}_m\right\} - \mathrm{Im}\left\{\tilde{h}_m\right\}\right), & \chi_m^{(+)} < \chi_m^{(-)} \end{cases}$$

where

$$\begin{aligned} \chi_m^{(+)} &= \left|\mathrm{Re}\left\{\tilde{h}_m\right\} + \mathrm{Im}\left\{\tilde{h}_m\right\}\right| \\ \chi_m^{(-)} &= \left|\mathrm{Re}\left\{\tilde{h}_m\right\} - \mathrm{Im}\left\{\tilde{h}_m\right\}\right| \end{aligned}$$

and

$$\begin{aligned} m &= \arg\max_p \left\{\max\left\{\left|\mathrm{Re}\left\{\tilde{h}_p\right\} + \mathrm{Im}\left\{\tilde{h}_p\right\}\right|, \left|\mathrm{Re}\left\{\tilde{h}_p\right\} - \mathrm{Im}\left\{\tilde{h}_p\right\}\right|\right\}\right\} \\ &= \arg\max_p \left\{\left|\mathrm{Re}\left\{\tilde{h}_p\right\}\right| + \left|\mathrm{Im}\left\{\tilde{h}_p\right\}\right|\right\} \end{aligned} \quad (4.3.20)$$

or equivalently,

$$t_i^* = h_i * \theta_i^* = \begin{cases} \delta(i+m)/\left(\mathrm{Re}\left\{\tilde{h}_m\right\} + \mathrm{Im}\left\{\tilde{h}_m\right\}\right), & \chi_m^{(+)} > \chi_m^{(-)} \\ j\delta(i+m)/\left(\mathrm{Re}\left\{\tilde{h}_m\right\} - \mathrm{Im}\left\{\tilde{h}_m\right\}\right), & \chi_m^{(+)} < \chi_m^{(-)} \end{cases}$$

which implies that

$$\begin{aligned} \theta_i^* &= \begin{cases} \tilde{h}_{i+m}/(\mathrm{Re}\left\{\tilde{h}_m\right\} + \mathrm{Im}\left\{\tilde{h}_m\right\}), & \chi_m^{(+)} > \chi_m^{(-)} \\ j\tilde{h}_{i+m}/(\mathrm{Re}\left\{\tilde{h}_m\right\} - \mathrm{Im}\left\{\tilde{h}_m\right\}), & \chi_m^{(+)} < \chi_m^{(-)} \end{cases} \\ &= \frac{\tilde{h}_{i+m} e^{jk\pi/2}}{\left|\mathrm{Re}\left\{\tilde{h}_m\right\}\right| + \left|\mathrm{Im}\left\{\tilde{h}_m\right\}\right|} \end{aligned}$$

where,

$$k = \begin{cases} 0, & \text{whenever } \mathrm{Re}\left\{\tilde{h}_m\right\} > 0 \text{ and } \mathrm{Im}\left\{\tilde{h}_m\right\} > 0 \\ 1, & \text{whenever } \mathrm{Re}\left\{\tilde{h}_m\right\} > 0 \text{ and } \mathrm{Im}\left\{\tilde{h}_m\right\} < 0 \\ 2, & \text{whenever } \mathrm{Re}\left\{\tilde{h}_m\right\} < 0 \text{ and } \mathrm{Im}\left\{\tilde{h}_m\right\} < 0 \\ 3, & \text{whenever } \mathrm{Re}\left\{\tilde{h}_m\right\} < 0 \text{ and } \mathrm{Im}\left\{\tilde{h}_m\right\} > 0 \end{cases}$$

∎

Remarks

1. The proof relies on the particular form of the cost in (4.3.18) that allows the cost to be viewed as an l_1 norm of a purely real sequence consisting of the real and imaginary components of the complex t-space parameters. Lemma 4.2 exploits this observation and is able to use the corresponding lemma for real sequences Lemma 4.1.

2. The particular form of the cost given by (4.3.18) is valid if both Assumption 2.1 holds and the symbol constellation is *square* (see Definition 4.1).

3. The proof cannot be used in those cases where the constellation is not square. For example, the 8-PSK and 32-QAM constellations fail to be square because they do not have any maximal corner symbols. In these cases, the following condition exists:

$$\begin{aligned}\mathcal{J}(t) &= \max_{\{a_k\}}\left|\sum_{i=-\infty}^{\infty}\operatorname{Re}\{t_i\}\operatorname{Re}\{a_{k-i}\}-\operatorname{Im}\{t_i\}\operatorname{Im}\{a_{k-i}\}\right| \\ &\quad +\max_{\{a_k\}}\left|\sum_{i=-\infty}^{\infty}\operatorname{Re}\{t_i\}\operatorname{Im}\{a_{k-i}\}+\operatorname{Re}\{a_{k-i}\}\operatorname{Im}\{t_i\}\right| \\ &\leq 2M\sum_{i=-\infty}^{\infty}|\operatorname{Re}\{t_i\}|+|\operatorname{Im}\{t_i\}|\end{aligned}$$

Therefore the explicit form of the cost in the t-space (4.3.18) *cannot* be used in the case of nonsquare constellations to formulate an equivalent l_1 type optimisation problem that is readily solved. This presents a major difficulty in generalising our admissibility result (Theorem 4.3) for the maximum complex deviation cost of (4.3.14) or (4.3.16) to any practical QAM constellation.

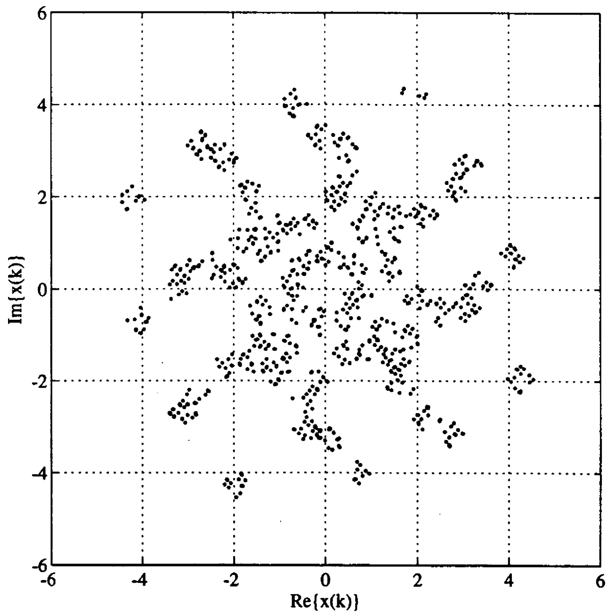
4.4 Summary

Much of this chapter has been spurred by the development of an *admissible* algorithm for real PAM systems proposed in [29, 24, 26]. Unique to this development is the new criterion for equalisation adopted, namely, *equalisation without gain identification*

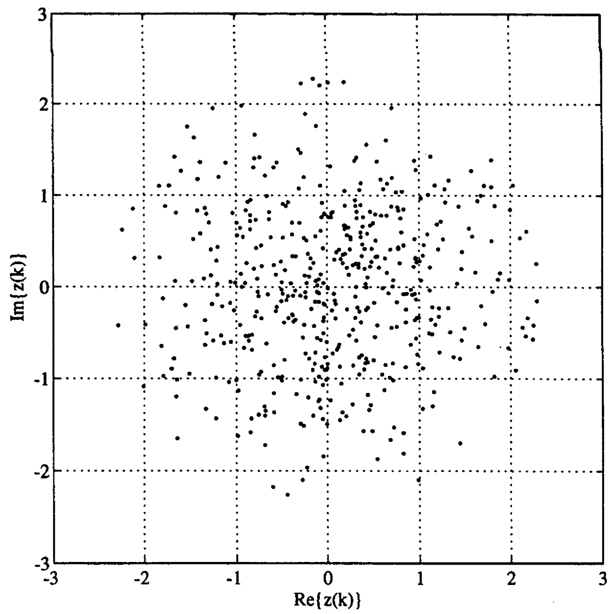
(EWGI). Here, we have documented the preliminary work on the design and analysis of an EWGI algorithm suitable for QAM systems using a convex cost with a linear constraint on the equaliser parameters. We presented the work as a series of generalisations of the real convex cost and single-parameter constraint proposed in [29, 24, 26]. In the process, we introduced a cost design philosophy that followed from the EWGI approach and subsequently used it to justify the original work of [29, 24, 26] for real PAM systems and that of [34] for complex QAM systems. Our philosophy considers the issues relevant to the design of a suitable cost and constraint for QAM transmission such as the problem of the phase offset of the equaliser output. These issues supplement those that must be considered for real PAM systems. We also presented a simple method that permits a more straightforward analysis of the admissibility of an algorithm. The method involved expressing both the cost and constraint as explicit functions in the combined channel-equaliser space or *total parameter space* (t-space). This allows the problem of linearly-constrained cost minimisation to be viewed as a conventional optimisation problem. Viewing the problem in the t-space also has the advantage that an ideal zero-ISI equaliser parameter setting has the simple form of the Kronecker delta sequence.

Our first generalisation of the convex cost and single-parameter constraint of [29, 24, 26] was proven inadmissible. From this naive generalisation, we identified the importance of designing the parameter constraint to allow the parameters to have some degree of phase freedom. Using our developed philosophy we motivated the next generalisation which was in fact proposed by [34]. It adopted a maximum complex deviation cost and a single-parameter constraint that allowed for phase freedom. We also provided a proof of its admissibility by formulating a t-space view of the constrained minimisation of the maximum complex deviation cost.

Despite its usefulness, the t-space analysis method assumes that the QAM constellation is *square* and that all subsequences of the symbol sequence are probable. Thus, our tool cannot be applied without modification to the case of nonsquare QAM constellations and in some channel coding scenarios where certain subsequences are prevented from occurring. Also the phenomenon of nonuniqueness was not discussed in a general light as it is attributed to the constraint and we have only considered specific single-parameter constraints here. In the next chapter we will further generalise our results and interpretation of nonunique minimisation behaviour by proposing an algorithm that adopts a maximum complex deviation cost and a *general* linear constraint on the equaliser parameters that is suitable for all practical QAM constellations.



a)



b)

Figure 4-3: 4-PSK, channel of Figure 4-1. a) Channel output. b) Equaliser output after cost is minimised.

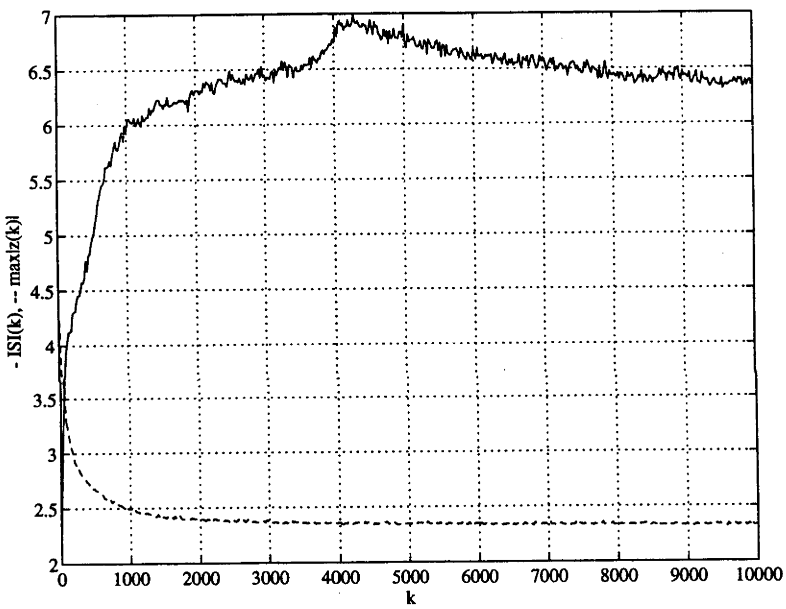
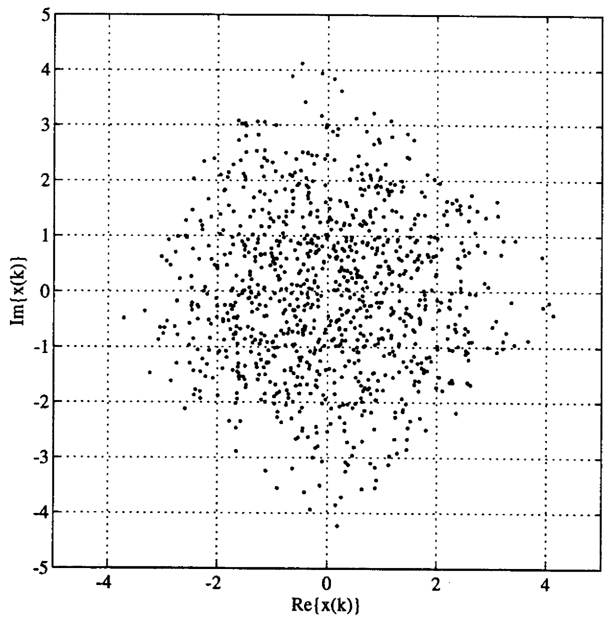
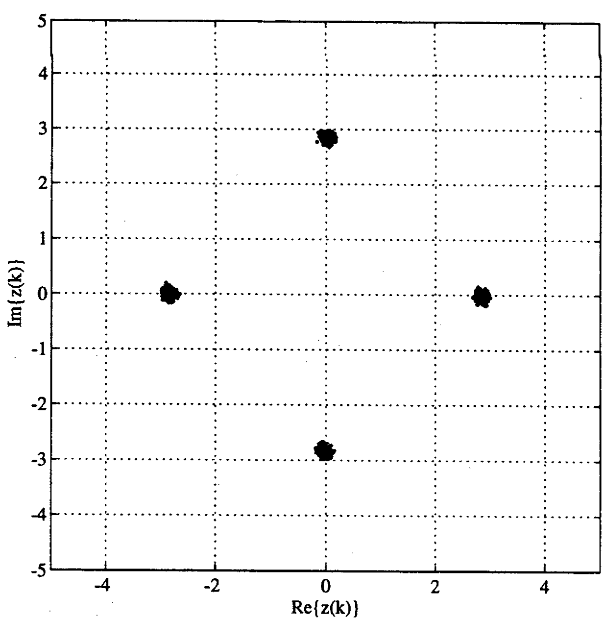


Figure 4-4: ISI and cost evolution. 4-PSK, channel of Figure 4-1.



a)



b)

Figure 4-5: 4-PSK, channel of (4.2.55). a) Channel output. b) Equaliser output after cost is minimised.

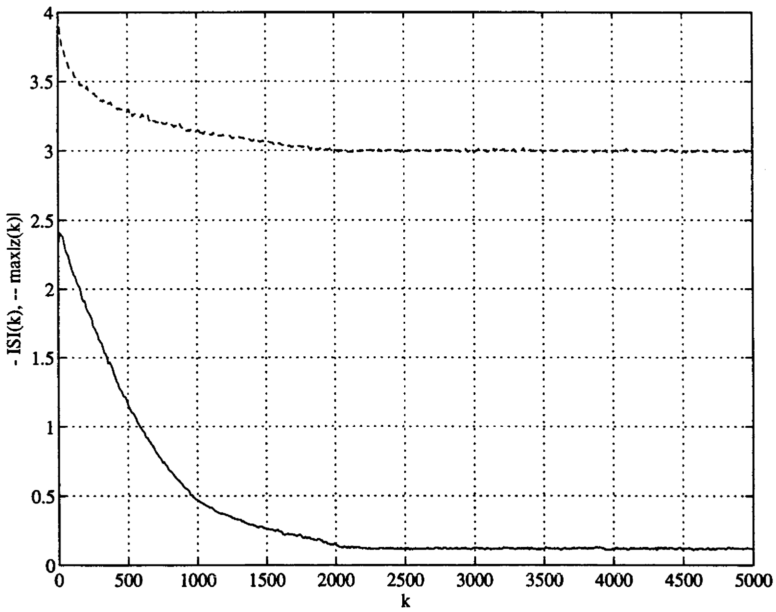


Figure 4-6: ISI and Cost evolution. 4-PSK, channel of (4.2.55).

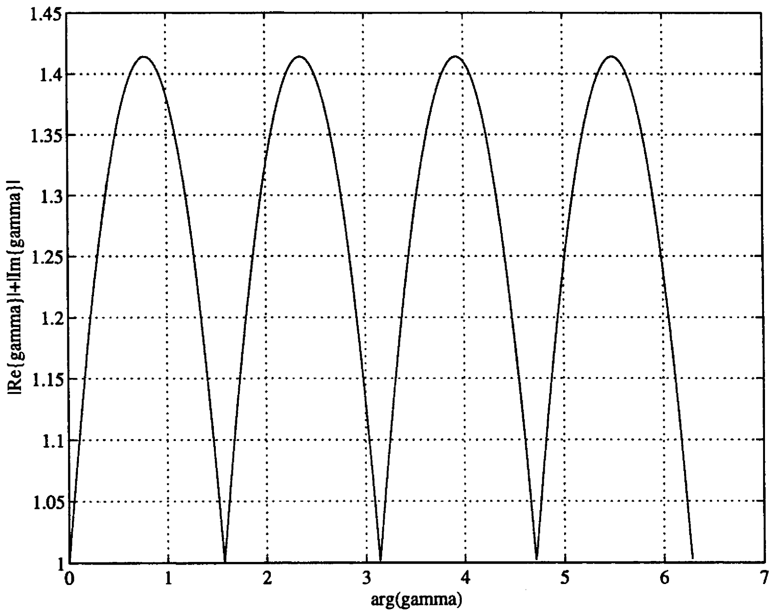


Figure 4-7: Plot of $\mathcal{D}(\gamma_\Delta)$ vs $\arg(\gamma_\Delta)$. $|\gamma_\Delta| = 1$.

Chapter 5

Maximum Complex Deviation Cost with General Linear Parameter Constraint

In this chapter, we document the final generalisation of the *equalisation without gain identification (EWGI)* algorithm proposed by [29, 24, 26] to a form suitable for QAM transmission transmission systems. From our earlier generalisations of the convex cost and linear (single) equaliser parameter constraint in Chapter 4, we introduced the idea of viewing the linearly-constrained cost minimisation problem in the *combined channel-equaliser parameter space* or *t-space*. This rendered an explicit form of the problem that could be interpreted as a simple l_1-type optimisation problem. Using this t-space interpretation we showed in a straightforward manner that a convex maximum complex deviation cost with a linear (single) parameter constraint [34] is *admissible*. However, a *limitation* to this approach is the necessary assumption that the symbol constellation be *square*. Without it the cost cannot be expressed in its simple l_1 form. Furthermore, we identified that the *nonuniqueness phenomenon*, an artefact of EWGI algorithms, is due to *particular*, *i.e.*, nongeneric combinations of the channel inverse impulse response and constraint. However, the phenomenon could not be understood in a general light because we assumed a specific (single-parameter) form of the linear constraint.

Our final generalisation here couples the *maximum complex deviation cost* of Chapter 4, Section 4.3 with a *general linear constraint* on the equaliser parameters. The highlight of this chapter is the introduction of a novel *geometric framework* whereby we use methods from *convex analysis* to pose the general linearly-constrained cost minimisation problem as one where an intersection is achieved between two geometric objects in the t-space, namely, a *cost level surface* (boundary of a convex polytope) and a *constraint hyperplane.* The former object is defined by the cost function and QAM constellation whilst the latter by the linear constraint sequence and channel impulse response. Unlike in Chapter 4, the framework assumes an almost *arbitrary* QAM constellation. Our geometric view gives an intuitive understanding of the cost minimisation problem and allows us to readily prove the *admissibility* of the algorithm. The nonuniqueness phenomenon is also generally understood by looking at particular degenerate orientations between the cost level surface and the constraint hyperplane.

In Section 5.1, we formally present our blind equalisation algorithm based on the maximum complex deviation cost and a general linear equaliser parameter constraint. In Section 5.2, we formulate our *geometric framework* by defining the abstract notions of a cost level surface and a constraint hyperplane in the t-space and then determining their properties. Based on our geometric framework, we state our main blind equalisation convergence (admissibility) result in Section 5.3 and provide a succinct proof. We also give a simple geometric interpretation of the mechanism of nonuniqueness. In Section 5.4 we apply our geometric interpretation to study nonunique minimisation behaviour of the cost and constraint of [34]. We also demonstrate our understanding by constructing a channel that produces such behaviour. In Section 5.5 we introduce a *prefilter formulation* of a general linear constraint on the equaliser parameters that establishes a *canonical* representation of a general linearly constrained equaliser. It also illuminates the role that the linear constraint may have on achieving close to ideal convergence behaviour. Section 5.6 deals with the issue of correlation in the channel input symbols and its effect on the convergence of the equaliser. In Section 5.7 we present one possible implementation of our blind equalisation algorithm and briefly compare its complexity to the *Constant Modulus Algorithm (CMA)* [11]. The effect of noise in the channel output on the convergence of the equaliser is briefly discussed in Section 5.8. We end this chapter with various simulations of our algorithms on a number of channels which demonstrate the convergence of the equaliser in the unique and nonunique minimisation cases as well as the case where channel noise is present.

5.1 Proposed Globally Convergent Algorithm

Consider a stable, complex, linear, causal, and possibly mixed-phase channel with impulse response $\{h_k\}$. The channel input symbol sequence $\{a_k\}$ is assumed stationary and comprised of symbols from an almost arbitrary QAM constellation $\mathcal{A}$ that satisifies Assumption 2.1 and Assumption 2.5 in Chapter 2. The channel output sequence $\{x_k\}$ can be expressed as

$$x_k = \sum_{i=0}^{\infty} h_i a_{k-i}, \quad \forall\, k. \tag{5.1.1}$$

A *linear transversal equaliser* with parameters $\{\theta_i\}$ is placed in cascade with the channel. The equaliser output is given by

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i x_{k-i}, \quad \forall\, k \tag{5.1.2}$$

or alternatively

$$z_k = \sum_{i=-\infty}^{\infty} t_i a_{k-i}, \quad \forall\, k \tag{5.1.3}$$

where $\{t_k\}$ denote the *total impulse response parameters* given by

$$t_k = \sum_{i=-\infty}^{\infty} \theta_i h_{k-i}, \quad \forall\, k. \tag{5.1.4}$$

The blind equalisation algorithm that we propose to update the equaliser parameters performs a (complex) stochastic gradient descent of the *maximum complex deviation cost* under a *general linear constraint* on the equaliser parameters, namely,

$$J(\theta) = \max_{\{a_k\}} \mathrm{Re}\,\{z_k\} \quad \left(= \max_{\{a_k\}} |\mathrm{Re}\,\{z_k\}|\right) \tag{5.1.5}$$

subject to

$$\sum_{i=-\infty}^{\infty} \mathrm{Re}\,\{\bar{c}_{-i}\theta_i\} = 1, \quad c_i \in \mathbb{C}, \tag{5.1.6}$$

where $\bar{c}_i$ denotes the complex conjugate of the constraint parameter c_i. We take the right hand side of (5.1.6) to be unity without loss of generality. The maximum complex deviation cost is equivalent to its absolute value form as indicated by (5.1.5) because of Assumption 2.1 and Assumption 2.5. This is indicated by expanding (5.1.5) using

(5.1.3) and then applying the assumptions, *viz.*,

$$\begin{aligned}
\mathcal{J}(\theta) &= \max_{\{a_k\}} \mathrm{Re}\,\{z_k\} = \max_{\{a_k\}} \sum_{i=-\infty}^{\infty} \mathrm{Re}\,\{t_i a_{k-i}\} \\
&= \sum_{i=-\infty}^{\infty} \max_{a_{k-i}\in\mathcal{A}} \mathrm{Re}\,\{t_i a_{k-i}\} && (5.1.7) \\
&= \sum_{i=-\infty}^{\infty} \max_{a_{k-i}\in\mathcal{A}} |\mathrm{Re}\,\{t_i a_{k-i}\}\,| \\
&= \max_{\{a_k\}} |\mathrm{Re}\,\{z_k\}|\,. && (5.1.8)
\end{aligned}$$

We use the first form of the maximum complex deviation given by (5.1.5) because it will facilitate the later developments in this chapter. Note that the constraint (5.1.6) is a linear one in terms of the real and imaginary parts of the equaliser parameters, that is,

$$\sum_{i=-\infty}^{\infty} \mathrm{Re}\,\{\bar{c}_{-i}\theta_i\} = \sum_{i=-\infty}^{\infty} \mathrm{Re}\,\{c_{-i}\}\,\mathrm{Re}\,\{\theta_i\} + \mathrm{Im}\,\{c_i\}\,\mathrm{Im}\,\{\theta_i\} = 1. \qquad (5.1.9)$$

The complex stochastic gradient descent is given by

$$\theta_i(k+1) = \theta_i(k) - \mu\, \frac{\partial}{\partial\bar{\theta}_i} J(\theta)\bigg|_{\theta=\theta(k)}, \quad i\in\mathbb{Z},\ \mu\in\mathbb{R} \qquad (5.1.10)$$

where μ is the update step-size parameter. Details about the implementation of our algorithm in practice will be treated later in Section 5.7.

5.2 Geometric Framework

In this section, we develop an abstract geometric framework in which the cost function $\mathcal{J}(\theta)$ and QAM symbol constellation $\mathcal{A}$ define a geometric object in the t-space which we call a *cost level surface.* It is parametrised by the value of the cost function and contains all t-space parameter settings that produce that value. We show that the cost level surface is the boundary of a closed and bounded set. This set takes the form of a *convex polytope* which is characterised by its *elementary boundary points* or *vertices.* The size of the polytope is determined by the cost value. We also define the notion of a general hyperplane in the t-space and the idea of increasing the size of the polytope (and hence the value of the cost) until it "*touches*" or *intersects* the hyperplane. The theorem of this section establishes the result that the intersection set must contain at least one elementary boundary point or vertex of the polytope. This abstract notion of the polytope touching the hyperplane will be interpreted later in Section 5.3 as the solution to the linearly-constrained cost minimisation problem when we consider that the linear constraint sequence and channel inverse define a hyperplane in the t-space.

5.2.1 Geometric Concepts

In this section, we introduce the concepts that are necessary in the development of our geometric view of our blind equalisation problem.

We consider the t-space as a particular *Banach space* $\mathcal{T}$ with generic element denoted by $t \in \mathcal{T}$. Specifically, we consider the space of double-sided infinite-length complex sequences with generic elements $\{t_i\}$, $t_i \in \mathbb{C}$, $i \in \mathbb{Z}$, equipped with the l_1 norm

$$\|t\|_1 \triangleq \sum_{i=-\infty}^{\infty} |t_i|. \tag{5.2.1}$$

We assume that the notion of a *compact subset* of $\mathcal{T}$ is familiar, meaning (as a consequence of the Banach space being a special case of a metric space) that such a compact subset is necessarily *bounded* and *closed.*

Next we require two related definitions pertaining to the definition of a *hyperplane* in the t-space.

Definition 5.1 (Hyperplane) *A hyperplane $\mathcal{H}$ excluding the origin is a subset of $\mathcal{T}$ and is given by*

$$\mathcal{H} \triangleq \left\{ t \in \mathcal{T} : \sum_{i=-\infty}^{\infty} Re\{\bar{\xi}_{-i} t_i\} = 1 \right\}$$

where $\{\xi_{-i}\}$ is a fixed complex sequence representing the "normal" to the hyperplane.

Definition 5.2 (Half-space) *The half-space $\mathcal{H}^+$ excluding the origin and derived from the hyperplane $\mathcal{H}$ is given by*

$$\mathcal{H}^+ \triangleq \left\{ t \in \mathcal{T} : \sum_{i=-\infty}^{\infty} Re\{\bar{\xi}_{-i} t_i\} > 1 \right\}.$$

It is important to recognise that both definitions *exclude the origin* in the t-space. The reason for this will be made evident in the development of the geometric framework to follow. Also, note that the summation in the definition of $\mathcal{H}$ can be viewed as a summation involving alternately the real and imaginary components of t as indicated by the expansion

$$\sum_{i=-\infty}^{\infty} \mathrm{Re}\{\bar{\xi}_{-i}t_i\} = \sum_{i=-\infty}^{\infty} \mathrm{Re}\{\xi_{-i}\}\,\mathrm{Re}\{t_i\} + \mathrm{Im}\{\xi_{-i}\}\,\mathrm{Im}\{t_i\} = 1 \tag{5.2.2}$$

and, hence, accommodates the case where the elements of $\mathcal{T}$ are double-sided *real* sequences.

Furthermore, there is a one-to-one correspondence between the equaliser parameters $\{\theta_i\}$ and the t-space parameters $\{t_i\}$ as indicated by (5.1.4). Therefore, we will may equivalently denote the value of the cost for a given equaliser parameter setting $\mathcal{J}(\theta)$ in terms of the corresponding t-space parameters, namely, $\mathcal{J}(t)$.

Finally, we define two subsets in the t-space that contain all t-space sequences whose corresponding cost values $\mathcal{J}(t)$ are either the same or are bounded by the same value. Recall that the cost $\mathcal{J}(t)$ that we adopt is defined by (5.1.5).

Definition 5.3 *The set of all t-space sequences whose corresponding cost values $\mathcal{J}(t)$ are bounded by some positive real number α is defined as*

$$\mathcal{P}_\alpha \triangleq \{t \in \mathcal{S} \mid \mathcal{J}(t) \leq \alpha,\ \alpha \in (0,\ \infty)\}.$$

The boundary of $\mathcal{P}_\alpha$ can be thought of as *level surfaces* of the cost function (when the cost is *equal* to α). Formally,

Definition 5.4 (Cost Level Surface at α) *The cost level surface at α is defined as*

$$\partial\mathcal{P}_\alpha \triangleq \{t \in \mathcal{S} \mid \mathcal{J}(t) = \alpha,\ \alpha \in (0,\ \infty)\}.$$

The cost function $\mathcal{J}(\cdot)$ in (5.1.5) involves a maximisation over all sequences $\{a_k\}$ where $a_k \in \mathcal{A},\ \forall\ k$. Therefore, both $\mathcal{P}_\alpha$ and the cost level surface at α depend on the geometry of the complex QAM constellation $\mathcal{A}$.

5.2.2 Cost Level Surface as a Boundary of a Convex Polytope

In this section, we determine the basic properties of $\mathcal{P}_\alpha$ and its boundary or *cost level surface* $\partial\mathcal{P}_\alpha$. In particular we establish that $\mathcal{P}_\alpha$ (for a fixed finite α) is bounded in the l_1 norm. We also make the significant observation that $\mathcal{P}_\alpha$ is a *convex polytope* that is characterised by its *elementary boundary points* or *vertices.*

The following lemmas establish the boundedness and convexity of the set $\mathcal{P}_\alpha$.

Lemma 5.1 Boundedness *Under Assumption 2.5, $\mathcal{P}_\alpha$ is bounded and contains the origin.*

Proof: Consider the following analysis on the cost function expression:

$$\begin{aligned} J(t) &= \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \mathrm{Re}\{t_i a_{k-i}\} \\ &\geq \sum_{i=-\infty}^{\infty} \epsilon |t_i|, \quad \epsilon > 0 \\ &= \epsilon \|t\|_1 \end{aligned}$$

where we have used (5.1.7) and Assumption 2.5 of Chapter 2. Hence for $\mathcal{P}_\alpha$ where $J(t) \leq \alpha$ we can infer

$$\|t\|_1 < \frac{\alpha}{\epsilon}.$$

Furthermore, from the cost function expression in the t-space, $J(t) = 0$ when $t = \{\ldots, 0, 0, 0, \ldots\}$ so any $\mathcal{P}_\alpha$, $0 < \alpha < \infty$, contains the origin. ∎

Lemma 5.2 Convexity *$\mathcal{P}_\alpha$ is convex.*

Proof: Consider

$$t \triangleq (1-\lambda)t^{(1)} + \lambda t^{(2)}, \quad \lambda \in [0,1], \ t^{(1)}, t^{(2)} \in \mathcal{P}_\alpha.$$

Then,

$$\begin{aligned} J(t) &= \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \mathrm{Re}\{t_i a_{k-i}\} \\ &= \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \left[(1-\lambda)\mathrm{Re}\left\{t_i^{(1)} a_{k-i}\right\} + \lambda \mathrm{Re}\left\{t_i^{(2)} a_{k-i}\right\}\right] \\ &\leq \sum_{i=-\infty}^{\infty} (1-\lambda) \max_{a_{k-i}} \mathrm{Re}\left\{t_i^{(1)} a_{k-i}\right\} + \lambda \max_{a_{k-i}} \mathrm{Re}\left\{t_i^{(2)} a_{k-i}\right\} \\ &= (1-\lambda) \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \mathrm{Re}\left\{t_i^{(1)} a_{k-i}\right\} + \lambda \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \mathrm{Re}\left\{t_i^{(2)} a_{k-i}\right\} = \alpha. \end{aligned}$$

Therefore, $\mathcal{P}_\alpha$ is a convex set. ∎

The following important lemma, corollary, and definition establish that the cost level surface at $\partial\mathcal{P}_\alpha$ can be characterised by sequences of a special form called *elementary* *boundary points* that also belong to the cost level surface. Specifically, Lemma 5.3 develops the result that any sequence t belonging to the cost level surface can be expressed as some convex combination of a subset of elementary boundary points. Corollary 5.1 establishes that if a sequence t of the cost level surface can be expressed as a convex combination of a particular subset of elementary boundary points then *any* convex combination of those elementary boundary points will yield a sequence of the cost level surface. Given the properties established here and the boundedness and convexity of $\mathcal{P}_\alpha$ established by Lemma 5.1 and Lemma 5.2, we show that $\mathcal{P}_\alpha$ is a *convex polytope*.

Lemma 5.3 Boundary Property *Any $t \in \partial\mathcal{P}_\alpha$ can be expressed as a convex combination of sequences of the form*

$$v \in \partial\mathcal{P}_\alpha \text{ such that } v_r = \begin{cases} \beta_i, & r = i \\ 0, & \text{otherwise} \end{cases}$$

for some $i \in \mathbb{Z}$ and $\beta_i \in \mathbb{C}$.

Proof: Our cost has the following form in the t-space.

$$J(t) = \max_{\{a_k\}} \sum_{i=-\infty}^{\infty} \operatorname{Re}\{t_i a_{k-i}\} \tag{5.2.3}$$

Under Assumption 2.1, *i.e.*, all sequences have nonzero probability, then the maximisation over all sequences can be rewritten as a sum of scalar maximisation problems, *viz.*,

$$J(t) = \sum_{i=-\infty}^{\infty} \max_{a_{k-i}} \operatorname{Re}\{t_i a_{k-i}\}. \tag{5.2.4}$$

Let

$$\alpha^{(i)} \triangleq \max_{a \in \mathcal{A}} \operatorname{Re}\{t_i a\} \tag{5.2.5}$$

whenever $t \in \partial\mathcal{P}_\alpha$. Then from (5.2.4) we have

$$\sum_{i=-\infty}^{\infty} \alpha^{(i)} = \alpha. \tag{5.2.6}$$

Further, by Assumption 2.5, $\alpha^{(i)} \geq 0$ for all $i \in \mathbb{Z}$. Then we can give a simpler expression for the boundary $\partial\mathcal{P}_\alpha$ using (5.2.4). Multiplying and dividing by $\alpha^{(i)}/\alpha$, we obtain

$$\partial\mathcal{P}_\alpha \triangleq \left\{ t \in \mathcal{S} \text{ such that } \sum_{i=-\infty}^{\infty} \left(\frac{\alpha^{(i)}}{\alpha}\right) \max_{a_{k-i}} \operatorname{Re}\left\{ \left(\frac{\alpha t_i}{\alpha^{(i)}}\right) a_{k-i} \right\} = \alpha \right\}. \tag{5.2.7}$$

If we let $\lambda_i \triangleq \alpha^{(i)}/\alpha$, then

$$\sum_{i=-\infty}^{\infty} \lambda_i \max_{a_{k-i}} \mathrm{Re}\left\{ \left(\frac{\alpha t_i}{\alpha^{(i)}}\right) a_{k-i} \right\} = \alpha. \tag{5.2.8}$$

Recalling the definition of $\alpha^{(i)}$, we note

$$\sum_{i=-\infty}^{\infty} \lambda_i = 1, \quad \lambda_i \geq 0. \tag{5.2.9}$$

The form of (5.2.8) indicates that any $t \in \partial\mathcal{P}_\alpha$ can be expressed as a convex combination of sequences in $\partial\mathcal{P}_\alpha$ (*i.e.*, have cost α) of the form

$$v_r = \begin{cases} \alpha t_i/\alpha^{(i)}, & r = i \\ 0, & \text{otherwise.} \end{cases} \tag{5.2.10}$$

∎

We see from the proof of Lemma 5.3 that any $t \in \partial\mathcal{P}_\alpha$ can be expressed as a particular convex combination of a subset of sequences of $\partial\mathcal{P}_\alpha$ that are of the special form given by (5.2.10). Note that t defines a *particular* subset of sequences of $\partial\mathcal{P}_\alpha$. The following corollary to Lemma 5.3 establishes that *any* convex combination of this subset of sequences will yield a sequence of $\partial\mathcal{P}_\alpha$.

Corollary 5.1 *Given any $t \in \partial\mathcal{P}_\alpha$, the convex hull of sequences*

$$v_r = \begin{cases} \alpha t_i/\alpha^{(i)}, & r = i \\ 0, & \textit{otherwise,} \end{cases} \tag{5.2.11}$$

where $\alpha^{(i)}$ is defined by (5.2.5) in Lemma 5.3, is a subset of $\partial\mathcal{P}_\alpha$.

Proof: In (5.2.8) we have that

$$\max_{a_{k-i}} \mathrm{Re}\left\{ \left(\frac{\alpha t_i}{\alpha^{(i)}}\right) a_{k-i} \right\} = \alpha \tag{5.2.12}$$

and, therefore, $J(t)$ simplifies to

$$J(t) = \alpha \sum_{i=-\infty}^{\infty} \lambda_i = \alpha. \tag{5.2.13}$$

This result implies that the sequence $\{\lambda_i\}$ can be replaced by any sequence $\{\lambda_i'\}$, $\lambda_i' \geq 0$, such that

$$\sum_{i=-\infty}^{\infty} \lambda_i' = 1 \tag{5.2.14}$$

without affecting the value of the cost. ∎

Lemma 5.3 and Corollary 5.1 establish that the cost level surface $\partial\mathcal{P}_\alpha$ can be characterised by a subset of sequences of $\partial\mathcal{P}_\alpha$ of the simple form given by (5.2.10). Accordingly, we make the following definition:

Definition 5.5 (Elementary Boundary Point) *A point, $v \in \partial\mathcal{P}_\alpha$ is an elementary boundary point of $\mathcal{P}_\alpha$ if it is of the form*

$$v_i = \begin{cases} \beta, & i = r \\ 0, & \textit{otherwise.} \end{cases}$$

for some $r \in \mathbb{Z}$ and $\beta \in \mathbb{C}$.

Given the simple characterisation of $\partial\mathcal{P}_\alpha$ by its elementary boundary points and the fact that $\mathcal{P}_\alpha$ is a bounded convex set, we say that $\mathcal{P}_\alpha$ is a *convex polytope.* Accordingly the elementary boundary points can be interpreted as the more familiar notion of the *vertices* of the convex polytope. Lemma 5.3 and Corollary 5.1 can also be interpreted as establishing that any point on the cost level surface must belong to an *edge* or *face* of the cost level surface. The notion that an edge or face (or higher dimensional analogue) is defined by a particular subset of the vertices of the convex polytope is also supported.

5.2.3 Cost Polytope Touching a Hyperplane

In the previous sections, we defined two important geometric objects in the t-space $\mathcal{T}$. The first object is a *general hyperplane* that excludes the origin (Definition 5.1) and whose orientation is determined by the normal sequence $\{\xi_{-i}\}$. The second is a *cost level surface* that is determined by the cost function and the geometry of the complex QAM constellation $\mathcal{A}$. We established that the cost level surface is the boundary of a *convex polytope* that contains the origin and is characterised by *elementary boundary points* or *vertices.*

Here we note that the size of the cost polytope $\mathcal{P}_\alpha$ is determined by α. We introduce the idea of inflating the cost polytope about the origin (by increasing α) until the polytope "*touches*" or intersects the hyperplane. We are guaranteed that there will be some finite nonzero size of the polytope $\alpha = \alpha^* \neq 0$ for which touching will occur because the hyperplane does not contain the origin. This touching condition is formalised by the following definition:

Definition 5.6 ($\mathcal{P}_\alpha$-$\mathcal{H}$ Touching) *$\mathcal{P}_\alpha$ is said to "touch" the hyperplane $\mathcal{H}$ when $\alpha = \alpha^*$ such that*

$$\mathcal{P}_{\alpha^*} \cap \mathcal{H} \neq \emptyset, \text{ and } \mathcal{P}_{\alpha^*} \cap \mathcal{H}^+ = \emptyset,$$

i.e.,

$$\alpha^* = \arg\min_{\alpha} \left\{ \mathcal{P}_\alpha \cap \mathcal{H} \neq \emptyset \right\}.$$

The following lemma states the intuitive notion that the set of intersection between the cost polytope $\mathcal{P}_{\alpha^*}$ and the hyperplane must be a subset of the boundary of the cost polytope, that is, a subset of the cost level surface $\partial\mathcal{P}_{\alpha^*}$. It is subsequently required by Theorem 5.1.

Lemma 5.4 *Let $\mathcal{P}_{\alpha^*}$ and the hyperplane $\mathcal{H}$ touch. Then $\mathcal{P}_{\alpha^*} \cap \mathcal{H} \subset \partial\mathcal{P}_{\alpha^*}$.*

Proof: Let $t \in \mathcal{P}_{\alpha^*} \cap \mathcal{H}$. Then either (i) $J(t) < \alpha^*$ or (ii) $J(t) = \alpha^*$. If (i) is true, then $J(t) = \alpha'$, i.e., $t \in \mathcal{P}_{\alpha'}$ for some $\alpha' < \alpha^*$. This contradicts the fact that α^* is the minimum such α for which the conditions hold. Hence (ii) is true $\Rightarrow t \in \partial\mathcal{P}_{\alpha^*}$. ∎

The next theorem is crucial to the geometric view of the linearly-constrained cost minimisation problem developed in the next section. It simply states that when the cost polytope and the hyperplane touch, the intersection set must contain at least one elementary boundary point or vertex. This is an intuitive result as we can envision the polytope touching the hyperplane on a point (a single vertex), an edge (the convex hull of two vertices) or a face (the convex hull of more than two vertices) of the polytope. The type of the intersection set depends on the orientation of the hyperplane or, in other words, on its defining normal sequence $\{\xi_{-i}\}$.

Theorem 5.1 *Let $\mathcal{P}_{\alpha^*}$ and the hyperplane $\mathcal{H}$ touch. Then $\mathcal{P}_{\alpha^*} \cap \mathcal{H}$ contains at least one elementary boundary point.*

Proof: Let $t \in \mathcal{P}_{\alpha^*} \cap \mathcal{H}$. Then, by Lemma 5.4, $t \in \partial\mathcal{P}_{\alpha^*}$. Hence, by Lemma 5.3, t

can be expressed as a convex combination of sequences $v^{(i)} \in \partial\mathcal{P}_{\alpha^*}$ of the form

$$v_r^{(i)} = \begin{cases} \beta_i, & r = i \\ 0, & \text{otherwise} \end{cases}$$

for $i \in \mathcal{I} \subset \mathbb{Z}$ and $\beta_i \in \mathbb{C}$. Let the set $\mathcal{L}$ be the convex hull of $\mathcal{V} = \{v^{(i)} \mid i \in \mathcal{I}\}$ and, therefore, $t \in \mathcal{L}$. By Corollary 5.1, $\mathcal{L} \subset \partial\mathcal{P}_{\alpha^*}$. Also, $\mathcal{L} \cap \mathcal{P}_{\alpha^*} \cap \mathcal{H} \neq \emptyset$ which implies one of two possibilities: (i) $\mathcal{L} \subset \mathcal{P}_{\alpha^*} \cap \mathcal{H}$ or (ii) $\mathcal{L} \not\subset \mathcal{P}_{\alpha^*} \cap \mathcal{H}$. Condition (ii) $\Rightarrow \exists\, t' \in \mathcal{L} \subset \mathcal{H}^+ \Rightarrow \mathcal{P}_{\alpha^*} \cap \mathcal{H}^+ \neq \emptyset$ which is in contradiction to the condition for touching hence condition (i) is true. Since $\mathcal{V} \subset \mathcal{L}$ then condition (i) $\Rightarrow v^{(i)} \in \mathcal{P}_{\alpha^*} \cap \mathcal{H},\ \forall\, i \in \mathcal{I}$. ∎

5.3 Geometric View of Linearly-Constrained Cost Minimisation

The geometric framework established in Section 5.2 provides us with the abstract notion of a *cost level surface* in the t-space. The cost level surface $\partial\mathcal{P}_\alpha$ defines the boundary of a convex polytope in the t-space whose size is determined by α. It is characterised by its *elementary boundary points* or *vertices*. The framework introduced the notion of expanding the polytope (and hence increasing the value of the cost α) until it intersects a *general hyperplane* in the t-space. The important result of that section is Theorem 5.1 which establishes that the intersection set must contain at least one vertex. The result agrees with the intuitive notion that the polytope must touch the hyperplane on a vertex, an edge or a face of the polytope.

Here we show that the channel inverse and linear parameter constraint sequence can be formulated as a *constraint hyperplane* in the t-space of the form given by Definition 5.1. We also make the observation that an elementary boundary point or *vertex* corresponds to a *zero-ISI equaliser parameter setting.* We then interpret the abstract notion of touching or intersecting with the constraint hyperplane as corresponding to finding the minimum cost $\alpha = \alpha^*$ under the linear parameter constraint. We apply Theorem 5.1 of Section 5.2 to establish that the intersection set, and hence the set of cost-minimising t-space parameter settings that meet the constraint, must contain at least one t-space parameter setting that corresponds to a zero-ISI equaliser parameter setting. This forms the basis of our subsequent blind equalisation convergence result.

5.3.1 Linear Constraint as a Hyperplane

Here we show that the channel inverse $\{\widetilde{h}_i\}$ and the linear parameter constraint sequence $\{c_i\}$ define a hyperplane in the t-space of the form given by Definition 5.1. Starting from the linear equaliser parameter constraint given by (5.1.6), we we may reexpress it using our convolution notation as follows:

$$\sum_{i=-\infty}^{\infty} \operatorname{Re}\{\bar{c}_{-i}\theta_i\} = \operatorname{Re}\{\bar{c}_0 * \theta_0\} = 1. \tag{5.3.1}$$

Next, we recall the following identity which relates the equaliser parameters to the equivalent t-space parameters:

$$\theta_k = \widetilde{h}_k * t_k, \quad \forall\, k. \tag{5.3.2}$$

Using (5.3.2) to express θ_k in terms of the t-space parameters t_k in (5.3.1) we have

$$\begin{aligned} \operatorname{Re}\{\bar{c}_0 * \theta_0\} &= \operatorname{Re}\left\{\bar{c}_0 * \widetilde{h}_0 * t_0\right\} \\ &= \operatorname{Re}\{\bar{\xi}_0 * t_0\} \\ &= \sum_{i=-\infty}^{\infty} \operatorname{Re}\{\bar{\xi}_{-i}\, t_i\} = 1 \end{aligned} \tag{5.3.3}$$

where

$$\bar{\xi} = \bar{c} * \widetilde{h}. \tag{5.3.4}$$

Note that the *constraint hyperplane* of (5.3.3) is precisely in the form of the hyperplane defined by Definition 5.1. The orientation of the constraint hyperplane is governed by the normal sequence $\{\xi_{-i}\}$ whose complex conjugate is the convolution of the complex conjugate of the linear parameter constraint sequence $\{c_i\}$ and the channel inverse impulse response $\{\widetilde{h}_i\}$ as indicated by (5.3.4).

With our geometric interpretation of the linear parameter constraint of (5.1.6) as a constraint hyperplane in the t-space $\mathcal{T}$, we may reinterpret the abstract notion of the cost polytope touching the constraint hyperplane. Namely, the process of inflating the cost polytope $\mathcal{P}_\alpha$ (increasing α) until it touches the constraint hyperplane, *i.e.*, $\alpha = \alpha^*$, can be interpreted as increasing the cost value α until the linear parameter constraint is satisfied. This corresponds to finding the minimum α^* of the linearly-constrained cost function. The intersection set in the t-space defines a corresponding set of equaliser parameter settings that simultaneously minimises the cost and satisfies the constraint. As we have already noted by applying Theorem 5.1, this intersection set can be an elementary boundary point or an edge or face of $\mathcal{P}_\alpha$. which are defined by two or more elementary boundary points. In the next section, we note that our notion of the elementary boundary point or vertex corresponds to an ideal equaliser parameter setting and subsequently derive a blind equalisation convergence result.

5.3.2 Blind Equalisation Convergence Result

Let us reconsider the form of an elementary boundary point $v \in \partial\mathcal{P}_\alpha$, that is,

$$v_i = \begin{cases} \beta, & i = r \\ 0, & \text{otherwise} \end{cases} \tag{5.3.5}$$

for some $r \in \mathbb{Z}$ and $\beta \in \mathbb{C}$. We now make the simple observation that an elementary boundary point or vertex of $\partial\mathcal{P}_\alpha$ corresponds to a zero-ISI equaliser parameter setting. Recall from our blind equalisation problem statement in Chapter 2, Section 2.5 that the following form of a t-space sequence corresponds to a zero-ISI equaliser parameter setting.

$$t_k = \gamma\delta_{k-\Delta}, \quad \forall\ k \tag{5.3.6}$$

where $\gamma \in \mathbb{C}$ and $\delta_{k-\Delta}$ is a Kronecker delta sequence with the single nonzero term at $k = \Delta$, $\Delta \in \mathbb{Z}$. Therefore, it can be seen that all elementary boundary points correspond to zero-ISI equaliser parameter settings where $\beta = \gamma$ and $r = \Delta$.

Coupling this result with those of the previous section we conclude that when the cost polytope touches the constraint hyperplane, the intersection set is either a single zero-ISI equaliser parameter setting, an edge defined by two zero-ISI settings or a face defined by more than two zero-ISI settings. From this we conclude that the convex cost function of (5.1.5) subject to the linear equaliser parameter constraint (5.1.6) is minimised by at least one zero-ISI equaliser parameter setting. Figure 5-1 illustrates the geometric view of the linearly-constrained cost minimisation problem.

Our conclusions are summarised by the following theorem. Given the geometric view established here, the subsequent proof is relatively short.

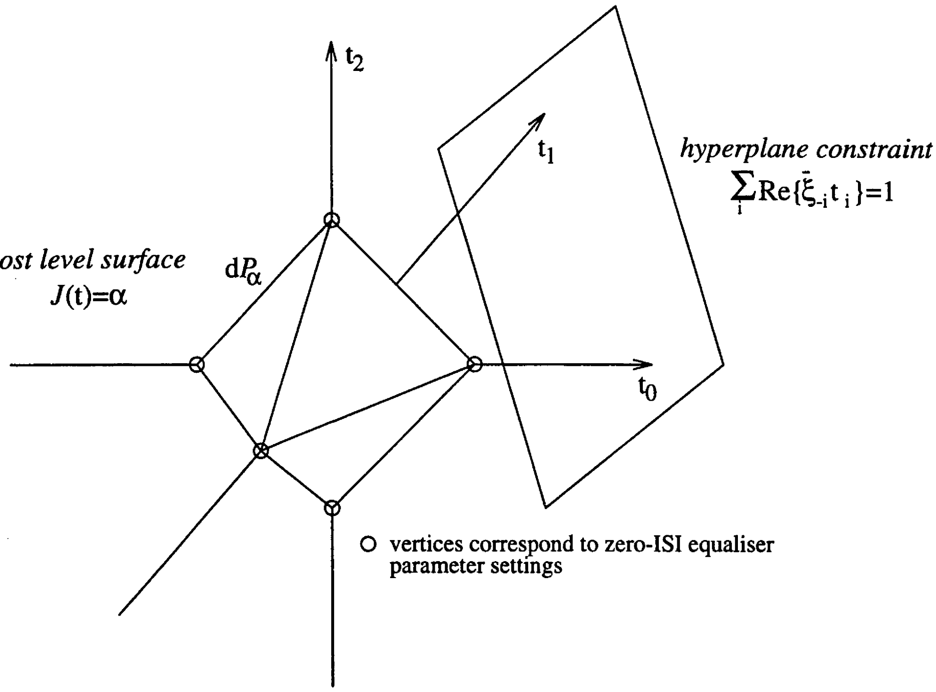


Figure 5-1: Geometric view of the linearly-constrained constrained cost minimisation problem.

Theorem 5.2 Convergence Result *Given a channel input of $\{a_k\}$ that satisfies Assumption 2.1 and consists of symbols from a complex QAM constellation $\mathcal{A}$ that satisfies Assumption 2.5. Consider an equaliser with parameters $\{\theta_i\}$ and output $\{z_k\}$. Then, the minimisation of the constrained blind equalisation algorithm criterion defined by*

$$J(\theta) = \max_{\{a_k\}} Re\{z_k\} \quad \left(= \max_{\{a_k\}} |Re\{z_k\}|\right)$$

subject to

$$\sum_{i=-\infty}^{\infty} Re\{\bar{c}_{-i}\theta_i\} = 1, \quad c_i \in \mathbb{C},$$

is achieved by an equaliser parameter setting corresponding to zero ISI.

Proof: Identify $\mathcal{P}_\alpha$ with the set $t \in \mathcal{T}$ such that $J(t) \leq \alpha$ and the constraint hyperplane with

$$\sum_{i=-\infty}^{\infty} \mathrm{Re}\{\bar{\xi}_{-i}\, t_i\} = 1$$

corresponding to the constraint on the equaliser parameters expressed in the t-space $\mathcal{T}$. Then by Theorem 5.1 the minimum cost is achieved by an elementary boundary point of $\mathcal{P}_\alpha$ when $\alpha = \alpha^*$. Since an elementary boundary point must be of the form

$$\{\ldots,\ 0,\ 0,\ \beta,\ 0,\ 0,\ \ldots\} \in \mathcal{T},$$

i.e., in the t-space, then this corresponds to zero ISI (since there is only one nonzero component in the rth position) and perfect equalisation up to a constant complex factor $t_r = \beta$. ∎

Remarks

1. The result is independent of the particular constraint chosen (provided the corresponding hyperplane does not pass through the origin). Similarly, if a fixed prefilter is used in the equaliser structure then again the result is valid.

2. Generically, the intersection set $\mathcal{P}_{\alpha^*} \cap \mathcal{H}$ is a single point in which case the minimum is achieved uniquely at an elementary boundary point of $\mathcal{P}_{\alpha^*}$ which corresponds to a zero-ISI equaliser parameter setting.

3. Nongenerically, it is possible that $\mathcal{P}_{\alpha^*} \cap \mathcal{H}$ is an extended compact set (e.g., an edge or face of $\mathcal{P}_\alpha$) in which case the minimum is achieved nonuniquely. Geometrically this corresponds to $\mathcal{P}_{\alpha^*}$ having a flat boundary parallel to the hyperplane.

4. The result is largely independent of the QAM constellation geometry (provided Assumption 2.1 and Assumption 2.5 hold). However, the geometry does affect the particular value of the complex gain factor β and its position r in the elementary boundary point sequence that minimises the cost.

Hence, generic minimisation is easily understood in terms of a single-point intersection between the cost boundary and the hyperplane. As this single point corresponds to a zero-ISI equaliser parameter setting, then this qualifies our algorithm as *admissible.* However, nongeneric minimisation, that is, minimisation achieved on a compact set, depends on a more intricate relationship between the cost boundary and a particular orientation of the hyperplane. In the next section we will construct an example of nonunique minimisation by orienting a constraint hyperplane parallel to a flat boundary of $\mathcal{P}_{\alpha^*}$ with a known simple structure.

5.4 Nonunique Minimisation Revisited

A globally convergent blind equalisation algorithm for QAM systems based on the minimisation of the maximum complex deviation cost function under a simple single-parameter constraint was introduced in [34] and examined in Chapter 4, namely,

$$J(\theta) = \max_{\{a_k\}} |\mathrm{Re}\{z_k\}| \tag{5.4.1}$$

subject to

$$\mathrm{Re}\{\theta_0\} + \mathrm{Im}\{\theta_0\} = 1. \tag{5.4.2}$$

Here, we revisit this cost and constraint and interpret them within our new geometric framework to gain further insight into the generic and nongeneric convergence properties.

5.4.1 Geometric View under the Square Constellation Case

As we have already noted, the cost (5.4.1) is equivalent to our maximum complex deviation cost given by (5.1.5). In [34], a *square* constellation (e.g. 4-QAM, 16-QAM) is assumed in order to simplify the analysis. Using this assumption and Assumption 2.1, the cost (5.4.1) may be simplified and expressed in the following convenient t-space form:

$$J(t) = M \sum_{i=-\infty}^{\infty} |\mathrm{Re}\{t_i\}| + |\mathrm{Im}\{t_i\}|, \quad M = \max_{\{a \in \mathcal{A}\}} \mathrm{Re}\{a\} = \max_{\{a \in \mathcal{A}\}} \mathrm{Im}\{a\}. \tag{5.4.3}$$

The single-parameter constraint (5.4.2) can be expressed in our framework in the form of the general linear constraint given by (5.1.6) where the constraint sequence, $\{c_i\}$, is

$$c_i = (1+j)\delta_i, \quad i \in \mathbb{Z} \tag{5.4.4}$$

where $j^2 = -1$. By (5.3.4), the constraint sequence $\{c_i\}$ and channel inverse impulse response $\{\widetilde{h}_i\}$ define the normal $\{\xi_i\}$ of the corresponding *constraint hyperplane* in the t-space:

$$\begin{aligned} \bar{\xi} &= \bar{c} * \widetilde{h} \\ &= (1-j)\widetilde{h}. \end{aligned} \tag{5.4.5}$$

Substituting (5.4.5) into (5.3.3) and expanding gives the following equation of the constraint hyperplane in the t-space.

$$\sum_{i=-\infty}^{\infty} \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} + \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Re}\left\{t_i\right\} + \left(\mathrm{Re}\left\{\tilde{h}_{-i}\right\} - \mathrm{Im}\left\{\tilde{h}_{-i}\right\}\right) \mathrm{Im}\left\{t_i\right\} = 1. \quad (5.4.6)$$

The *cost level surface* $\partial\mathcal{P}_\alpha$ can be constructed using (5.4.3) and is given by

$$\partial\mathcal{P}_\alpha = \left\{ t \in \mathcal{S} \,\middle|\, M \sum_{i=-\infty}^{\infty} |\mathrm{Re}\left\{t_i\right\}| + |\mathrm{Im}\left\{t_i\right\}| = \alpha \right\} \quad (5.4.7)$$

Equation (5.4.7) defines the boundary of an l_1 ball $\mathcal{P}_\alpha$ which is a convex polytope. This becomes evident if we treat the real and imaginary parts of t_i as separate real components in the space of real sequences. The size of the l_1 ball is determined by α.

Then, according to Theorem 5.1 and Theorem 5.2, if we inflate the l_1 ball (increase α) until it touches the constraint hyperplane (5.4.6), then this will occur at $\alpha = \alpha^*$. Upon touching, there exists at least one zero-ISI equaliser parameter setting with the t-space form, $v = \{\ldots, 0, \beta, 0, \ldots\} \in \partial\mathcal{P}_\alpha$, in the intersection set giving a minimum cost of

$$J(v) = M(|\mathrm{Re}\left\{\beta\right\}| + |\mathrm{Im}\left\{\beta\right\}|) = \alpha^*. \quad (5.4.8)$$

Since $\partial\mathcal{P}_{\alpha^*}$ is the boundary of an l_1 ball with vertices, edges, and faces, we may divide the nature of the intersection set into two cases. In the generic case, the l_1 ball $\mathcal{P}_{\alpha^*}$ will touch the constraint hyperplane at a vertex and hence a single zero-ISI equaliser parameter setting. In the nongeneric case, the l_1 ball will touch the hyperplane on either an edge or face, or, in other words, a compact set that is determined by the convex hull of two or more zero-ISI equaliser parameter settings.

5.4.2 A Degenerate Orientation of the Hyperplane

To illustrate the nongeneric case, consider two vertices $v^{(1)}, v^{(2)} \in \partial\mathcal{P}_{\alpha^*}$ of the form:

$$v_i^{(1)} = \begin{cases} \alpha, & i = r_1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow J(v^{(1)}) = \alpha \quad (5.4.9)$$

$$v_i^{(2)} = \begin{cases} \alpha, & i = r_2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow J(v^{(2)}) = \alpha \quad (5.4.10)$$

where $\alpha \in \mathbb{R}$, $r_1 \neq r_2$. A convex combination

$$v_\lambda = (1-\lambda)v^{(1)} + \lambda v^{(2)}, \quad 0 \leq \lambda \leq 1 \quad (5.4.11)$$

defines an edge of $\partial\mathcal{P}_\alpha$ as the following calculation shows:

$$\begin{aligned} J(v_\lambda) &= M\left[|\mathrm{Re}\{(1-\lambda)\alpha\}| + |\mathrm{Im}\{(1-\lambda)\alpha\}| + |\mathrm{Re}\{\lambda\alpha\}| + |\mathrm{Im}\{\lambda\alpha\}|\right] \\ &= (1-\lambda)\alpha + \lambda\alpha \\ &= \alpha. \end{aligned}$$

To achieve nonunique minimisation, we must orient the constraint hyperplane (5.4.6) such that the above edge is contained in the intersection set of the hyperplane and $\mathcal{P}_\alpha$ under a touching condition. Substituting (5.4.11) into (5.4.6) and simplifying we arrive at

$$\widetilde{h}'_{r_1}(1-\lambda)\alpha + \widetilde{h}'_{r_2}\lambda\alpha = 1 \tag{5.4.12}$$

where

$$\widetilde{h}'_r = \mathrm{Im}\left\{\widetilde{h}_{-r}\right\} + \mathrm{Re}\left\{\widetilde{h}_{-r}\right\}, \quad r \in \{r_1, r_2\}.$$

Thus, the constraint of (5.4.6) in this case reduces to a constraint on the elements of a transformed channel inverse sequence $\{\widetilde{h}'_i\}$. Solving for the cost value α we have

$$\alpha = \frac{1}{\widetilde{h}'_{r_1} + \lambda(\widetilde{h}'_{r_2} - \widetilde{h}'_{r_1})}. \tag{5.4.13}$$

Equation (5.4.13) makes explicit the conditions on $\{\widetilde{h}'_i\}$ sufficient to cause nonunique minimisation. First, α must be invariant to λ for $\lambda \in [0,1]$. This implies that to achieve nonunique minimisation,

$$\widetilde{h}'_{r_1} = \widetilde{h}'_{r_2} = \max_{i\in\mathbb{Z}} \widetilde{h}'_i > 0. \tag{5.4.14}$$

Equation (5.4.14) implicitly gives a condition of the channel inverse sequence that leads to nonunique minimisation under the single-tap constraint. An example of such a channel expressed in its inverse form is:

$$\widetilde{h}_i = \begin{cases} 0.8 + 0.2j, & i = 0 \\ 0.4 + 0.6j, & i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.4.15}$$

where, in this case, $r_1 = 0$ and $r_2 = 1$ and $\max_{i\in\mathbb{Z}} \widetilde{h}'_i = 1$. We will verify by simulation that the channel of (5.4.15) causes nonunique cost minimisation to occur in Section 5.9.

5.5 Prefilter Representation of Linear Constraint

Here we demonstrate the equivalence relations between a *general* linear constraint on the equaliser parameters and a system which uses a *fixed* linear prefilter prior to the equaliser which is subject to an equivalent constraint.

A prefilter equaliser cascade is shown in Fig.5-2 where the prefilter represents a fixed linear filter with impulse response $\{p_i\}$, $p_i \in \mathbb{C}$. We assume that the Z-transform of $\{p_i\}$ has no zeros on the unit circle. The equaliser parameters are constrained according to

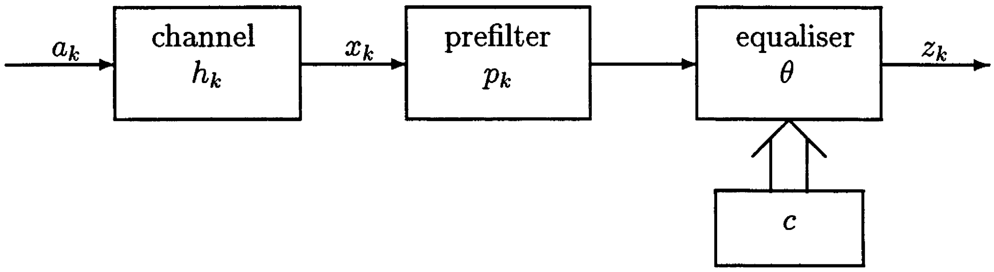


Figure 5-2: Prefilter-Equaliser Arrangement

(5.1.6) or expressed more succinctly in convolution notation, *i.e.*,

$$\mathrm{Re}\,\{\bar{c}_0 * \theta_0\} = 1. \tag{5.5.1}$$

At this point we remark that there is nothing particularly significant about taking the real part rather than the imaginary part or even more generally a linear combination of the real and imaginary parts in this expression for the constraint. The significant point is that it represents a projection onto one component (direction) of the two dimension (complex) space. This notion is degenerate for purely real systems in which case the taking the real part is superfluous.

To establish the equivalence result, let $\{\phi_i\}$ denote the impulse response of the cascade of the prefilter and equaliser, namely,

$$\phi_k = p_k * \theta_k, \quad \forall\ k. \tag{5.5.2}$$

The cascade can be interpreted as an *augmented equaliser* with parameters $\{\phi_i\}$. However, our linear equaliser parameter constraint in (5.5.1) is in terms of the original equaliser parameters $\{\theta_i\}$. To find an equivalent representation of the constraint in terms of the augmented equaliser parameters $\{\phi_i\}$ we apply a convolution to both sides of equation (5.5.2) with the *inverse* of the prefilter response which we denote by $\{\widetilde{p}_i\}$. This operation yields

$$\theta_k = \widetilde{p}_k * \phi_k, \quad \forall\ k \tag{5.5.3}$$

where we use the following general property of an inverse:

$$p_k * \widetilde{p}_k = \delta_k, \quad \forall\ k \tag{5.5.4}$$

where $\{\delta_k\}$ is the Kronecker delta sequence. Finally, we substitute the expression of θ_k in (5.5.3) into the linear parameter constraint equation of (5.5.1) to obtain

$$\mathrm{Re}\left\{(\bar{c}_0 * \widetilde{p}_0) * \phi_0\right\} = 1. \tag{5.5.5}$$

Figure 5-3 illustrates the new configuration of the channel and augmented equaliser. The systems of Figure 5-2 and Figure 5-3 represent *equivalent* systems in that the *total impulse responses* of both systems (between the channel input and equaliser output) are identical.

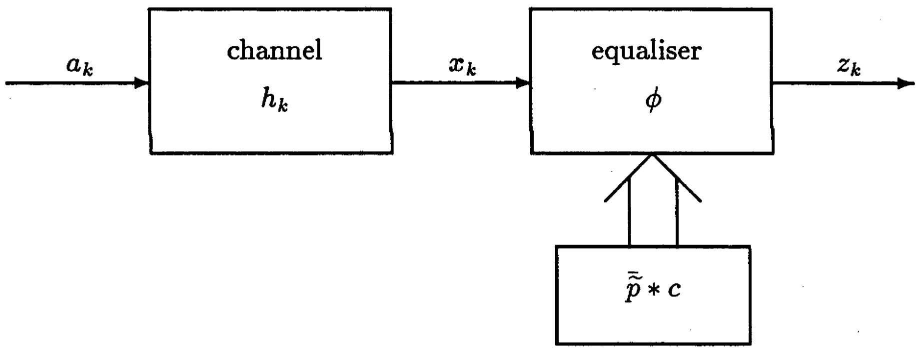


Figure 5-3: Equivalent Constraint Arrangement.

Furthermore, the effective linear constraints that the two systems impose on the total impulse response $\{t_k\}$ are identical. To see this, we refer to original prefilter equaliser configuration of Figure 5-2. The total impulse response is given by

$$t_k = h_k * p_k * \theta_k, \quad \forall\ k. \tag{5.5.6}$$

We may express the equaliser parameters $\{\theta_i\}$ in terms of the total impulse response by convolution of both sides with the inverses of the channel and prefilter responses, namely,

$$\theta_k = \widetilde{h}_k * \widetilde{p}_k * t_k, \quad \forall\ k. \tag{5.5.7}$$

The equaliser parameters given by (5.5.7) are constrained according to (5.5.1). Substituting (5.5.7) into (5.5.1) we obtain

$$\mathrm{Re}\left\{\left(\bar{c}_0 * \widetilde{h}_0 * \widetilde{p}_0\right) * t_0\right\} = 1. \tag{5.5.8}$$

The total impulse response of the configuration in Figure 5-3 is

$$t_k = h_k * \phi_k, \quad \forall\, k. \tag{5.5.9}$$

The augmented equaliser parameters can be expressed in terms of the total impulse response as follows:

$$\phi_k = \widetilde{h}_k * t_k, \quad \forall\, k. \tag{5.5.10}$$

The augmented equaliser parameters given by (5.5.10) are constrained according to (5.5.5). Substituting (5.5.10) into (5.5.5) we obtain

$$\mathrm{Re}\left\{\left(\bar{c}_0 * \widetilde{p}_0 * \widetilde{h}_0\right) * t_0\right\}. \tag{5.5.11}$$

Therefore, we have shown that the configurations of Figure 5-2 and Figure 5-3 impose the same constraint on the total impulse response parameters and therefore are equivalent representations.

We can show the equivalence between apparently different systems more conveniently by setting up the following notation:

$$[\,p; c] \Longleftrightarrow \text{prefilter } p, \text{ equaliser } \theta, \text{ subject to } \mathrm{Re}\{\bar{c}_0 * \theta_0\} = 1 \tag{5.5.12}$$

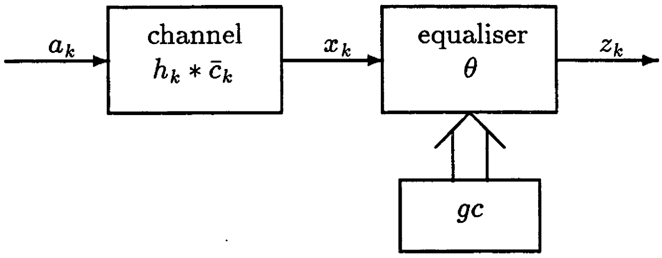
where p and c represent the prefilter impulse response $\{p_i\}$ and constraint sequence $\{c_i\}$. Therefore the equivalence established between Figure 5-2 and Figure 5-3 can be succinctly represented by

$$[\,h * p; c] \equiv [\,h; \bar{\widetilde{p}} * c]\,. \tag{5.5.13}$$

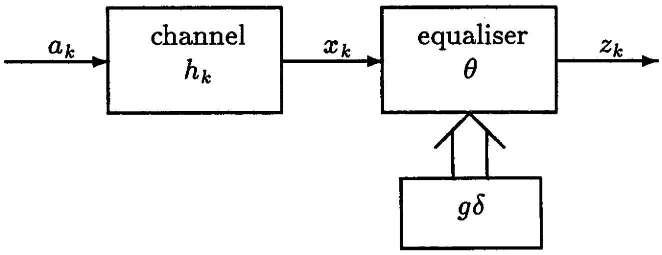
Using this notation we can simply derive other equivalent channel, prefilter and equaliser configurations. One important relation derived from (5.5.13) in effect provides the *canonical representation of a channel and constraint combination*:

$$[\,h; g\delta] \equiv [\,h; gc * \widetilde{c}] \equiv [\,h * \bar{c}; gc]\,, \quad g \in \mathbb{C}. \tag{5.5.14}$$

The relation (5.5.14) implies that, without loss of generality, the combination of a channel $\{h_k\}$ and single equaliser tap constraint $\{g\delta_k\}$ can be regarded equivalent to a combination of another channel $h'_k = h_k * \bar{\tilde{c}}_k$, $\forall\ k$ and a constraint $\{gc_k\}$ on the equaliser. The decomposition, $\delta_k = c_k * \tilde{c}_k$, is arbitrary and hence the channel/single-tap constraint combination can be considered a canonical representation of a class of equivalent channel/constraint combinations. To illustrate this relationship, Figure 5-4 gives an arbitrary channel/constraint combination and its corresponding canonical representation. Note that we have reverted to using θ to denote the parameters of the equivalent equaliser without loss of generality.



a)



b)

Figure 5-4: a) A channel/constraint combination where the sequence $\{c_k\}$ is arbitrary. b) The corresponding canonical representation.

Finally, consider the situation in Figure 5-2 and 5-3 when $c = g\delta$, a single-parameter constraint, and $p = \tilde{h}$. Then we have the following equivalence:

$$[\,h; g\bar{h}] \equiv [\,\delta; g\delta] \tag{5.5.15}$$

Equation (5.5.15) indicates that a channel-equaliser pair where the equaliser utilises a constraint of $g\bar{h}$ is equivalent to a fully equalised channel with an equaliser under a single parameter constraint. This implies that the equaliser need only zero all of its free parameters: a task that can be accomplished by any simpler even convex function such as a quadratic. The constraint of $g\bar{h}$ is thus “optimal” in some sense with the added benefit that nonuniqueness is avoided. This result forms the basis of an adaptive linear constraint strategy which is discussed in Chapter 6. Figure 5-5 illustrates a channel-equaliser combination with the "optimal" linear constraint on the equaliser parameters and the equivalent configuration where there is no ISI.

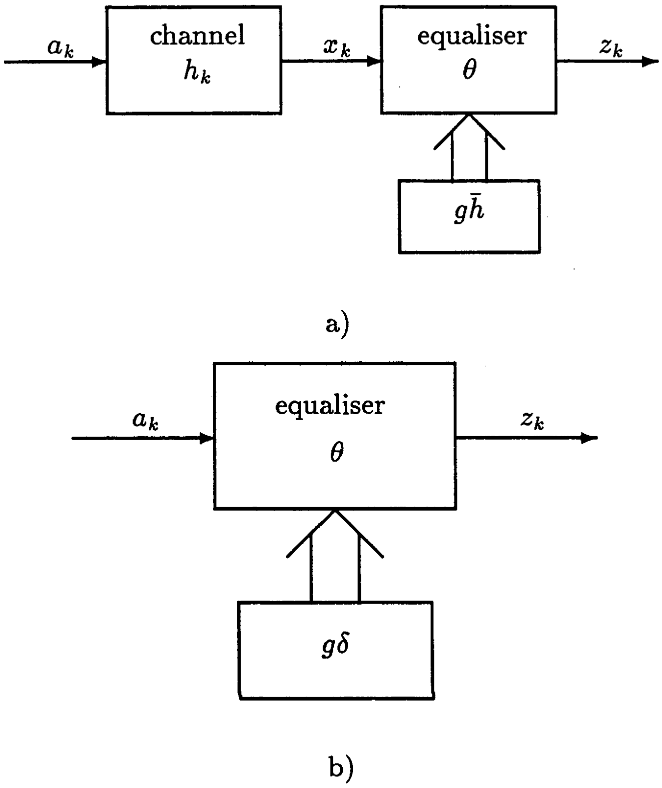


Figure 5-5: a) A channel and equaliser configuration with an "optimal" linear constraint on the equaliser parameters. b) An equivalent representation with no channel.

5.5.1 An Equivalent Non-Generic Channel/Constraint Combination

From the discussion in Section 5.5, the nongeneric channel/single-tap constraint combination constructed in Section 5.4.2 is a canonical representation of a class of nongeneric channel/constraint combinations. Using the equivalence relation given by (5.5.14), an arbitrary equivalent channel/constraint representation can be generated from this channel and single-tap constraint and an arbitrary constraint sequence $\{r_i\}$ as follows:

$$[\,h; (1+\jmath)\delta] \equiv [\,h; (1+\jmath)r * \tilde{r}] \equiv [\,h * \bar{r}; (1+\jmath)r]$$

and hence

$$h_i^{(\text{equiv})} = h_i * \bar{r}_i, \quad c_i^{(\text{equiv})} = (1+j)r_i. \tag{5.5.16}$$

In other words, we would expect the above channel/constraint pair to also cause nonunique minimisation behaviour. This will be confirmed by simulation in Section 5.9.

5.6 Effect of Symbol Correlation on Parameter Convergence

In our characterisation of the *cost level surface* in Section 5.2.2 we used Assumption 2.1 to prove Lemma 5.3. This lemma establishes the property that any point on the boundary of the cost level surface can be expressed as a convex combination of what we later define as *elementary boundary points* of the cost level surface. These elementary boundary points or *vertices* of the cost level surface correspond to zero-ISI equaliser parameter settings. Generally, the cost level surface will touch the hyperplane constraint at one of these points and this corresponds to global minimisation of the cost by a zero-ISI equaliser parameter setting. Up to this point, we have not yet considered the effect that violating Assumption 2.1 has on the boundary property of the cost level surface and hence the overall effect on the convergence of the equaliser parameters.

In this section, we look at a particular type of symbol correlation and study its effect on the convergence of blind equalisation algorithms which adopt a cost of the form of the maximum complex deviation cost of (5.1.5). Specifically, we consider the case where certain types of channel coding schemes, e.g., block coding, cause the absence of some symbol subsequences and therefore leads to a violation of Assumption 2.1.

To simplify our analysis we deal with a *real version* of the cost in (5.1.5) which is the same as that adopted in [29, 24, 26] for PAM systems. The analysis of the complex QAM case is expected to be similar. Previously introduced in Chapter 4, the real cost has the form

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k|. \tag{5.6.1}$$

Let us consider the simple case of a finitely parametrised equaliser with real parameters θ_i, $i = 0, \ldots, 3$. We assume that binary PAM symbols are input into a noiseless real channel with no ISI, *i.e.*, $x_k = a_k, \forall\, k$. Expressing the equaliser output $\{z_k\}$, in terms of the t-space parameters, we have

$$z_k = \sum_{i=0}^{3} t_i a_{k-i}, \quad \forall\ k \tag{5.6.2}$$

where, in our simple case, $t_i = \theta_i$, $\forall\ i$. Using (5.6.2), the cost of (5.6.1) can be reexpressed as

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k| = \max_{\{a_k\}} \left| \sum_{i=0}^{3} t_i a_{k-i} \right|. \tag{5.6.3}$$

We assume a linear equaliser parameter constraint of the form

$$\sum_{i=0}^{3} c_i \theta_i = \sum_{i=0}^{3} c_i t_i = 1. \tag{5.6.4}$$

where $\{c_i\}$ denotes the constraint sequence. Due to the absence of channel ISI, the form of (5.6.4) directly gives the constraint hyperplane in the t-space. The normal to the hyperplane is given by the constraint sequence $\{c_i\}$ and therefore it determines the orientation of the hyperplane.

In the case where Assumption 2.1 holds and the elements of the constraint sequence have distinct magnitudes, our cost takes the following form because of the freedom to choose among all symbol sequences of length four to maximise the summation in (5.6.3):

$$\mathcal{J}(\theta) = \mathcal{J}(t) = \sum_{i=0}^{3} |t_i|. \tag{5.6.5}$$

This corresponds to a cost level surface that is an l_1 ball in the t-space whose vertices correspond to zero-ISI equaliser equaliser settings.

Now suppose that channel coding prevents the following four-symbol sequences from occurring:

$$\{+1, +1, +1, +1\} \text{ and } \{-1, -1, -1, -1\}. \tag{5.6.6}$$

If we construct the level surface of the cost corresponding to $\mathcal{J}(\theta) = r$, then the two missing sequences correspond to the elimination of two faces of the original cost level surface (5.6.5) corresponding to

$$\sum_{i=0}^{3} t_i = r \text{ and } \sum_{i=0}^{3} t_i = -r. \tag{5.6.7}$$

In this case, the adjacent faces are extended out until they intersect. The resulting cost level surface is the boundary of the *convex hull* of an l_1 ball of radius r and two *additional vertices*, namely,

$$\left\{ \frac{+r}{2}, \frac{+r}{2}, \frac{+r}{2}, \frac{+r}{2} \right\} \text{ and } \left\{ \frac{-r}{2}, \frac{-r}{2}, \frac{-r}{2}, \frac{-r}{2} \right\}. \tag{5.6.8}$$

Note that the additional vertices clearly correspond to *closed-eye* equaliser parameter settings in the t-space. Therefore, if the constraint hyperplane is oriented such that touching occurs on one of these vertices, then this corresponds to global minimisation of the cost by a closed-eye equaliser parameter setting. We may force the hyperplane to touch the new cost level surface on one of these additional vertices by aligning the normal of the hyperplane $\{c_i\}$ to point to one of these vertices, *i.e.*,

$$c_i = 1 \text{ or } c_i = -1 \quad i = 0, \ldots, 3. \tag{5.6.9}$$

The corresponding output of the equaliser is

$$z_k = \pm r \sum_{i=0}^{3} a_{k-i}. \tag{5.6.10}$$

Figure 5-6 gives the equaliser output after 10000 iterations of the blind equalisation algorithm using the cost of (5.6.1) and binary PAM symbols with the two subsequences given by (5.6.6) omitted. The constraint sequence used was $c_i = 1$, $i = 0, \ldots, 3$. As can be seen from the three-level equaliser output, a closed-eye condition exists. The corresponding ISI and cost evolution is shown in Figure 5-7. The ISI is evaluated during the simulation according to (5.9.9) in Section 5.7. Note that the ISI curve reaches a maximum value of 3 at approximately 1500 iterations and remains at that level thereafter.

We have therefore constructed and simulated a simple example which demonstrates how channel coding that bars symbol subsequences can alter the ideal geometry of the cost level surface and introduce the potential for the minimisation of the cost to occur at a closed-eye equaliser parameter setting. Although this is an inadmissibility result, we make the following remarks which places this result in perspective.

Remarks

1. Coding will only potentially affect the form of the cost level surface when it bars symbol sequences of a length that is equal to or less than the (significant) length of the t-space sequence formed by the cascade of the channel and equaliser.

2. Even if the length of a barred symbol subsequence is less than that of the t-space sequence, it will only affect the cost level surface if it is a maximising subsequence (if its length equals the length of the t-space sequence) or is part

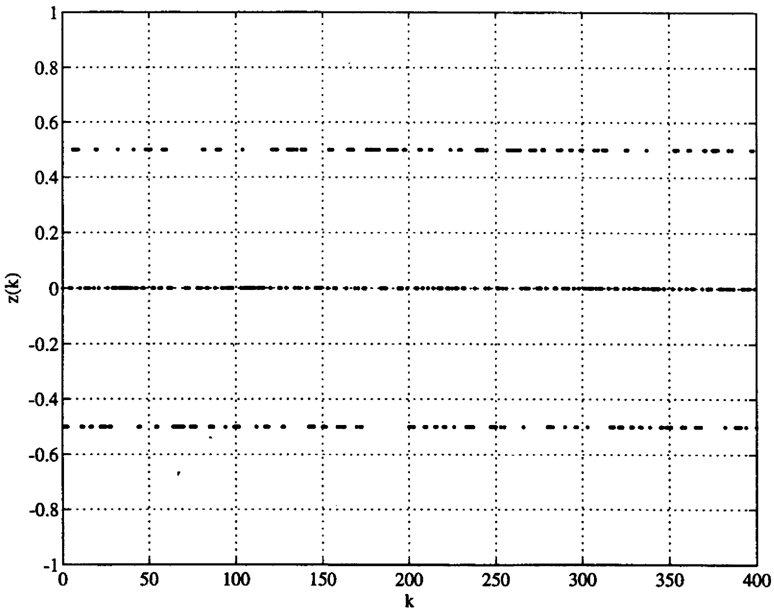


Figure 5-6: Equaliser output after 10000 iterations. Zero-ISI channel, binary PAM, $\mu = 0.001$. A window of 400 output samples is shown.

of a maximising subsequence (if its length is less than the length of the t-space sequence). For example, if multi-level PAM symbols (with a maximum symbol of M) are assumed for our construction, then maximising sequences must be of the form $\{\ldots, \pm M, \pm M, \ldots\}$. If all such maximising subsequences are permitted then barring other subsequences will not affect the geometry of the cost level surface.

3. Even under block coding, the total barring of specific symbol sequences may be rare considering that they may appear as sequences extending across block boundaries.

4. Global minimisation of the cost by a closed-eye equaliser parameter setting only occurs when the constraint hyperplane is oriented such that the corresponding closed-eye vertex touches the constraint hyperplane.

5. The negative effect of barring symbol subsequences is in principle not unique to our algorithm because source correlation has the potential to degrade *any* system identification procedure because it alters the *persistence of excitation* of the unknown system.

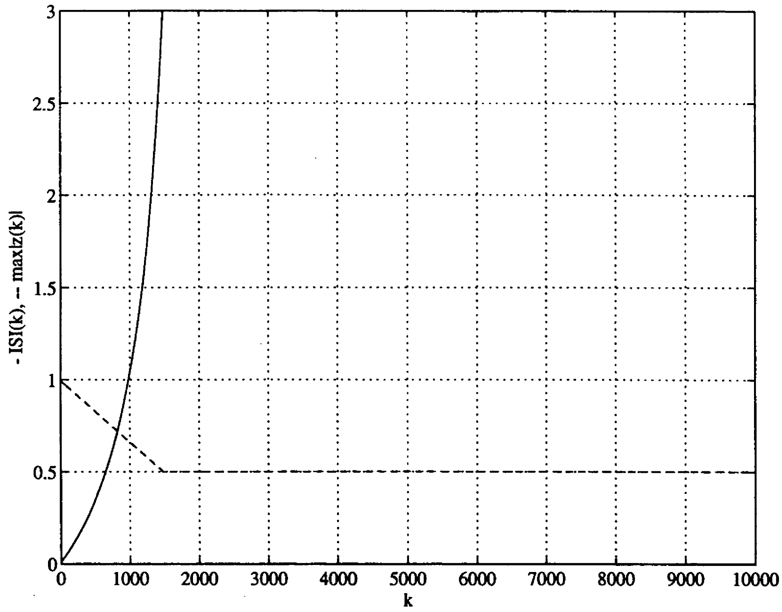


Figure 5-7: ISI and cost evolution. Zero-ISI channel, binary PAM, $\mu = 0.001$.

5.7 Algorithm Implementation

A buffer-based scheme for approximating the max($\cdot$) function of the cost in (5.1.5) is adopted in the implementation of our algorithm and is based on a method in [34]. A finite-length equaliser of length L is assumed sufficient to model the inverse of the channel. Here, it is convenient to represent $\boldsymbol{\theta}$ and related quantities as complex vectors in $\mathbb{C}^L$. The buffer-based scheme operates by maintaining a buffer of length B containing the regressor of the equaliser output and the corresponding regressor of the channel output of length $B + L - 1$. The cost in (5.1.5) is approximated by searching the equaliser output regressor for the value with the maximum real part. The details of the approximation are given below:

$$J(\boldsymbol{\theta}) \approx \mathrm{Re}\{z_p\} = \mathrm{Re}\left\{\boldsymbol{\theta}^H \boldsymbol{X}_p\right\} \tag{5.7.1}$$

where

$$\begin{aligned} \boldsymbol{\theta} &= [\ \theta_0 \ \ \theta_1 \ \ \ldots \ \ \theta_{L-1}\ ]^T, \quad \boldsymbol{\theta} \in \mathbb{C}^L \\ \boldsymbol{X}_p &= [\ x_p \ \ x_{p-1} \ \ \ldots \ \ x_{p-L+1}\ ]^T, \quad \boldsymbol{X}_p \in \mathbb{C}^L \end{aligned}$$

and

$$p = \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\{z_d\} = \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\left\{\boldsymbol{\theta}^H \boldsymbol{X}_d\right\}, \quad \mathcal{B}_k \triangleq \{k - B + 1, \ldots, k\}. \tag{5.7.2}$$

Unlike in [34], the adaptation of the parameters is dictated by a two-step process whereby the first step is a small gradient descent adaptation step and the second step is the projection of the adapted parameter vector onto the constraint vector. The gradient of the approximate cost is

$$\begin{aligned} \frac{\partial}{\partial \bar{\theta}_i} J(\boldsymbol{\theta}) &= \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re}\{\theta_i\}} J(\boldsymbol{\theta}) + j \frac{\partial}{\partial \mathrm{Im}\{\theta_i\}} J(\boldsymbol{\theta})\right) \\ &= \frac{1}{2}\left(\mathrm{Re}\{x_{p-i}\} + j\mathrm{Im}\{x_{p-i}\}\right) \\ &= x_{p-i}/2. \end{aligned} \tag{5.7.3}$$

The practical parameter update equation follows by substituting (5.7.3) into the stochastic gradient update equation (5.1.10) and letting the adaptation step-size μ absorb the factor of 1/2.

$$\theta_i(k+1) = \theta_i(k) - \mu x_{p-i}, \quad i \in \{0 \ldots L-1\}, \ \mu \in \mathbb{R}, \tag{5.7.4}$$

where p is determined from (5.7.2).

Frost's projection method [35] is adopted for the projection operation as it avoids parameter trajectories off of the constraint due to accumulated numerical errors. Since we are dealing with projections in $\mathbb{C}^L$, it is necessary to define a real-valued inner product that allows a projection in $\mathbb{C}^L$ to have a geometric interpretation as a projection in $\mathbb{R}^{2L}$ where the real and imaginary components are treated as pairs of real numbers [36]:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle \triangleq \mathrm{Re}\left\{\boldsymbol{A}^H \boldsymbol{B}\right\}, \quad \boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^L \tag{5.7.5}$$

$$= \sum_{i=0}^{L-1} \mathrm{Re}\{a_i\}\,\mathrm{Re}\{b_i\} + \mathrm{Im}\{a_i\}\,\mathrm{Im}\{b_i\} \tag{5.7.6}$$

where a_i and b_i are the ith elements of $\boldsymbol{A}$ and $\boldsymbol{B}$.

In the context of Frost's algorithm, it is convenient to represent the constraint of (5.5.1) in inner product form as

$$\langle \boldsymbol{C}, \boldsymbol{\theta} \rangle = 1 \tag{5.7.7}$$

where $\boldsymbol{C} \in \mathbb{C}^L$ contains the reversed finite-length constraint parameters

$$\boldsymbol{C} = \begin{bmatrix} c_{L-1} & \ldots & c_1 & c_0 \end{bmatrix}^T.$$

Frost's projection is as follows:

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k+1) - \mathrm{proj}_{\mathbf{C}}(\boldsymbol{\theta}(k+1)) - \frac{\boldsymbol{C}}{\langle \boldsymbol{C}, \boldsymbol{C} \rangle} \tag{5.7.8}$$

where $\mathrm{proj}_{\mathbf{C}}(\boldsymbol{X})$ denotes a "projection of $\boldsymbol{X}$ onto $\boldsymbol{C}$" namely

$$\mathrm{proj}_{\mathbf{C}}(\boldsymbol{X}) = \frac{\langle \boldsymbol{C}, \boldsymbol{X} \rangle}{\langle \boldsymbol{C}, \boldsymbol{C} \rangle} \boldsymbol{C} \tag{5.7.9}$$

Hence (5.7.8) expands to

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k+1) - (\langle \boldsymbol{C}, \boldsymbol{\theta}(k+1) \rangle - 1) \frac{\boldsymbol{C}}{\langle \boldsymbol{C}, \boldsymbol{C} \rangle}. \tag{5.7.10}$$

In summary, the two-step algorithm is given below:

The Implemented Algorithm

Step 1 (Unconstrained Gradient Descent Step)

$$\theta_i(k+1) = \theta_i(k) - \mu x_{p-i}, \quad i \in \{0 \dots L-1\}$$

where $\mu \in \mathbb{R}$ is the adaptation step-size and

$$p = \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\{z_d\} = \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\left\{\boldsymbol{\theta}^H \boldsymbol{X}_d\right\}, \quad \mathcal{B}_k \triangleq \{k - B + 1, \dots, k\}.$$

Step 2 (Frost's Projection onto Constraint)

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k+1) - (\langle \boldsymbol{C}, \boldsymbol{\theta}(k+1) \rangle - 1) \frac{\boldsymbol{C}}{\langle \boldsymbol{C}, \boldsymbol{C} \rangle}.$$

The computational requirements exceed that of the well-known *Constant Modulus Algorithm (CMA)* [11] due to the buffer-based implementation of the cost function in (5.7.2) and the additional projection step given by (5.7.10). Specifically, the update of Step 1 is of the same computational complexity as the CMA parameter update procedure. However, prior to this, p must be determined by computing and buffering B values of the equaliser output for a particular equaliser parameter setting and searching for the value in the buffer with the greatest real part. The projection in Step 2 requires further computations. Note also that $B + L - 1$ channel output values are required for each iteration of the algorithm as opposed to the single one required for each iteration of the CMA algorithm.

Note that our implementation requires a memory of $2B + L - 1$. As we shall see in the next section a reasonable value for B is 1000 for 4-QAM and 5000 for 32-QAM with the buffer size increasing with increasing QAM levels. However, this does not present too severe a requirement for most general DSP chips which typically have an addressing space of at least 64K.

For computational simplicity, it may be advantageous to implement a one-step linear constraint adaptation procedure that avoids the separate projection step by using the *General Sidelobe Canceller* model described in [37]. Further possible improvements to the implementation of the algorithm are suggested in Chapter 6.

5.8 Effect of Channel Noise on Parameter Convergence

In our characterisation of our linearly-constrained convex cost function and the resulting blind equaliser convergence theorem we have assumed a noiseless channel model. However, any practical implementation of a blind equaliser must possess a certain degree of robustness to system noise as it is unavoidable. In this section, we briefly examine the effect of an additive noise signal at the output of the channel. Figure 5-8 illustrates a channel and equaliser configuration with an additive channel output noise signal $\{\nu_k\}$.

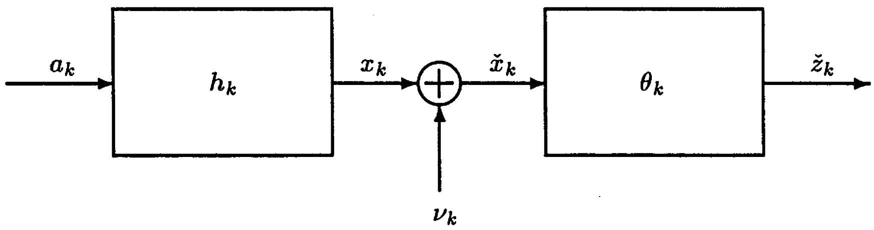


Figure 5-8: Simplified channel and equaliser configuration with an additive noise signal ν_k at the output of the channel.

The noisy channel output is given by

$$\check{x}_k = x_k + \nu_k. \tag{5.8.1}$$

A suitable stochastic model for the additive noise signal ν_k is a *zero mean Gaussian random process* with variance σ_ν^2. With the noise signal added, the equaliser output is

$$\check{z}_k = \sum_{i=-\infty}^{\infty} \theta_i(x_{k-i} + \nu_{k-i})$$

$$\begin{aligned} &= \sum_{i=-\infty}^{\infty} \theta_i x_{k-i} + \sum_{i=-\infty}^{\infty} \theta_i \nu_{k-i} \\ &= z_k + \sum_{i=-\infty}^{\infty} \theta_i \nu_{k-i} \end{aligned} \tag{5.8.2}$$

where z_k is the noise-free equaliser output.

To study the effect of the noise on the convergence of the equaliser parameters we recall the equaliser parameter update of (5.7.4) for the implementation of our algorithm in Section 5.7. Substituting the noisy channel output $\{\check{x}_k\}$ into (5.7.4) we obtain

$$\begin{aligned} \theta_i(k+1) &= \theta_i(k) - \mu \check{x}_{p-i} \\ &= \theta_i(k) - \mu(x_{p-i} + \nu_p) \end{aligned} \tag{5.8.3}$$

where

$$p = \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\left\{ \sum_{i=0}^{L-1} \theta_i \check{x}_{d-i} \right\}, \quad \mathcal{B}_k = k - B + 1, \ \ldots, \ k. \tag{5.8.4}$$

Note from (5.8.3) that there are two mechanisms whereby the noise may affect the parameter update process. The first is the explicit addition of the noise term $\mu\nu_p$ to the noiseless update term μx_{p-i}. However, if we examine the noisy update term averaged over the noise we obtain

$$\mathrm{E}\left[\mu(x_{p-i} + \nu_p)\right] = \mu \mathrm{E}\left[x_{p-i}\right] + \mu \mathrm{E}\left[\nu_p\right] = \mu x_{p-i}. \tag{5.8.5}$$

Equation (5.8.5) indicates that, *on average*, the parameter update term is the same as that in the noiseless case. The second mechanism is the corruption of the search for the maximum $\mathrm{Re}\{z_k\}$ in the equaliser output buffer implied by (5.8.4). Each value in the equaliser output buffer will have an additive noise component which, if sufficiently large, may cause an incorrect maximising index p to be chosen in the following sense:

$$\arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\left\{ \sum_{i=0}^{L-1} \theta_i \check{x}_{d-i} \right\} \neq \arg\max_{d \in \mathcal{B}_k} \mathrm{Re}\left\{ \sum_{i=0}^{L-1} \theta_i x_{d-i} \right\}. \tag{5.8.6}$$

Further analysis of the effect of the noise will involve formulating a suitable bound on the noise signal $\{\nu_k\}$ or σ_ν^2 relative to the noiseless received signal $\{x_k\}$ that establishes some upper limit on the probability of occurrence of the condition given by (5.8.6). Although, we do not present such an analysis here, we make the proposition that moderate received signal (noiseless) to noise ratios (SNR) will not cause ill-convergence of the equaliser parameters. Furthermore, we do not expect divergence of the equaliser parameters because the very nature of the parameter update is to reduce a measure of the maximum size of the equaliser output. Empirical support for these propositions is given in the next section where our algorithm is simulated for various levels of SNR.

5.9 Simulations

The simulations included in this section demonstrate the performance of the algorithm under various channel/constraint combinations. Specifically, the following four selected channel/constraint combinations are used to demonstrate unique and nonunique minimisation behaviour exhibited by our algorithm. Note that for Channel/Constraint 2 and 3, the channel is specified in its inverse form. Infinite-length channel impulse responses are approximated with finite-length channels of length K.

Channel/Constraint 1

$$H^{(1)}(z^{-1}) = e^{j\pi/4}\frac{0.7 - z^{-1}}{1 - 0.7z^{-1}}, \quad K = 10 \tag{5.9.1}$$

$$c_i^{(1)} = \begin{cases} 0.4e^{j\pi/3}, & i = -1 \\ e^{j\pi/3}, & i = 0 \\ -0.3e^{j\pi/3}, & i = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{5.9.2}$$

Channel/Constraint 2

$$\widetilde{h}_i^{(2)} = \begin{cases} 0.8 + 0.2j, & i = 0 \\ 0.6 + 0.4j, & i = 1 \\ 0, & \text{otherwise} \end{cases}, \quad K = 21 \tag{5.9.3}$$

$$c_i^{(2)} = \begin{cases} 1 + j, & i = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5.9.4}$$

Channel/Constraint 3

$$\widetilde{h}_i^{(3)} = \begin{cases} 0.8 + 0.2j, & i = 0 \\ 0.6 + 0.3j, & i = 1 \\ 0, & \text{otherwise} \end{cases}, \quad K = 21 \tag{5.9.5}$$

$$c_i^{(3)} = c_i^{(2)}. \tag{5.9.6}$$

Channel/Constraint 4

$$h_i^{(4)} = h_i^{(2)} * c_i^{*(1)}, \quad K = 23 \tag{5.9.7}$$

$$c_i^{(4)} = (1+j)c_i^{(1)}. \tag{5.9.8}$$

The channel of Channel/Constraint 1 is a first-order all-pass channel with an added rotation of $\pi/4$. The constraint has been arbitrarily chosen and consists of 3 nonzero coefficients. This combination was chosen to demonstrate the generic unique minimisation behaviour, that is, the case where the constraint hyperplane and the cost polytope touch at a single point in the equaliser parameter space. Figure 5-9 shows the typical channel output x_k and resulting equaliser output z_k of a 32-QAM simulation using this channel and constraint after 2000 iterations. Figure 5-10 gives the corresponding ISI and cost evolution.

The ISI measure used in all of the simulations is:

$$ISI(k) \triangleq \frac{\sum_{i=0}^{L+K-2} |t_i(k)|}{|t_p(k)|} - 1, \quad p = \arg\max_i |t_i(k)| \tag{5.9.9}$$

where $t_i(k) = h_i * \theta_i(k)$, $i = 0 \ldots L+K-2$.

Figure 5-11 to Figure 5-16 give the equaliser output and corresponding ISI and cost evolution for the 32-QAM simulation under signal-to-noise ratios (SNR) of 30 dB, 20 dB, and 10 dB. The SNR is calculated as the ratio of the power of the noise-free channel output $\{x_k\}$ and the power of the noise signal $\{\nu_k\}$ as folows:

$$\text{SNR} = \frac{\mathrm{E}\,[x_k^2]_{\text{est}}}{\mathrm{E}\,[\nu_k^2]_{\text{est}}}. \tag{5.9.10}$$

where the expectations $\mathrm{E}\,[\cdot]_{\text{est}}$ are estimated by averaging over a window of approximately 5000 samples. Note from the equaliser output plots the progressive degradation of the open-eye condition under a SNR of 30 dB to a closed-eye condition under a SNR of 10 dB.

To illustrate the case of of nonunique minimisation behaviour, Channel/Constraint 2 is the pathological channel/constraint developed in Section 5.4.2. Typical channel and equaliser outputs of a 4-QAM simulation are shown in Figure 5-17. Note that the equaliser output implies that its parameters have adapted to the geometric mean of the two possible zero-ISI equaliser parameter settings. Figure 5-18 gives the ISI and cost evolution. The behaviour between $k \approx 200 \ldots 1500$ clearly demonstrates that, while the cost is minimised, the ISI may increase as the parameters wander between the two possible open-eye solutions. The pertinent equaliser parameter trajectories are shown in Figure 5-19. Here, the final values of the parameters conform to the geometric mean of the two possible open-eye solutions.

Figure 5-20 to Figure 5-22 illustrate the result of slightly perturbing the channel of Channel/Constraint 2 to avoid the nonunique (nongeneric) minimisation behaviour in the 4-QAM simulation. Specifically, Channel/Constraint 3 is used in this demonstration and is identical to Channel/Constraint 2 with the exception that $\mathrm{Im}\left\{\widetilde{h}_0^{(3)}\right\} = 0.3$ and hence $\mathrm{Re}\left\{\widetilde{h}_0^{(3)}\right\} + \mathrm{Im}\left\{\widetilde{h}_0^{(3)}\right\} > \mathrm{Re}\left\{\widetilde{h}_0^{(3)}\right\} + \mathrm{Im}\left\{\widetilde{h}_0^{(3)}\right\}$. Note that the final values of the parameters indicated by Figure 5-22 approximately give the channel inverse.

The constraint of the pathological Channel/Constraint 2 combination is a single-tap linear constraint and, by the discussion in Section 5.5, is a canonical representation of an equivalence class of nonunique channel/constraint combinations. A transformed yet equivalent representation of Channel/Constraint 2 of the form (5.5.16) is given in Channel/Constraint 4. The equaliser output after 10000 iterations of a 4-QAM simulation using Channel/Constraint 4 is shown in Figure 5-23 and empirically confirms that combination's equivalence to the canonical representation.

5.10 Summary

Our final generalisation of the cost and constraint of [29, 24, 26] was documented here where we considered a convex maximum complex deviation cost subject to a general linear constraint on the equaliser parameters. We built upon the simple t-space analysis tool developed in Chapter 4 and established a more general framework in which the problem of constrained cost minimisation could be studied using a novel *geometric view* of the problem. The geometric view facilitates a more intuitive understanding of the problem in both the generic and nongeneric (nonuniqueness phenomenon) cases. Moreover, unlike our simple tool, this framework can accommodate an almost arbitrary QAM constellation. Obtaining the geometric view involves a translation of the problem into the t-space as before. However, this translation depends less on the particular form of the convex cost and QAM constellation used. Instead, two geometric objects corresponding to the cost and constraint are defined in the t-space, namely, a *cost level surface* and a *constraint hyperplane*. The orientation of the hyperplane was shown to depend on both the channel and the constraint. The minimisation problem is then viewed as one of scaling the size of the cost level surface object until it "*touches*" the constraint hyperplane. The scale at which touching occurs corresponds to the minimum value of the linearly-constrained cost and the intersection set determines the set of equaliser parameter settings that simultaneously conform to the constraint and minimise the cost. In this light, our most important result was to show that the cost level surface is the boundary of a *convex polytope* whose *elementary boundary points* or *vertices* correspond to zero-ISI equaliser parameter settings.

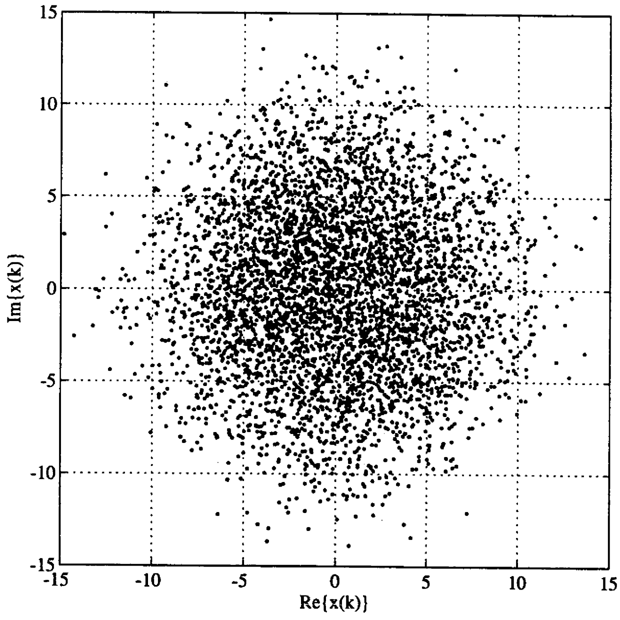
The generic situation is one where a vertex touches the constraint hyperplane which implies global minimisation of the cost by a *particular* zero-ISI equaliser parameter setting. This is our *global convergence result.* To contrast, nonunique minimisation is viewed as the nongeneric situation where the hyperplane is oriented in such a way that the polytope touches the hyperplane on an *edge* or *face.* This implies that all of the vertices (*i.e.*, zero-ISI equaliser parameter settings) that define the edge or face minimise the cost which is clearly a case of nonunique minimisation. With this view, it became clear that the nonuniqueness phenomenon is attributed to a *particular* combination of the channel and general constraint because it is this combination that determines the particular degenerate orientation of the hyperplane. Using the insight gained, we were able to construct an example of nonunique minimisation and verified the result with simulations.

Another powerful tool that we developed was the *prefilter formulation* of a general linear constraint on the equaliser parameters. This formulation establishes the equivalence between an equaliser with a general linear constraint and the cascade of a *prefilter* and an equaliser with a single parameter constraint. The equivalence stems from the fact that the t-space representations of the two configurations are identical. Furthermore, the channel-equaliser-single tap constraint combination is shown to be a *canonical representation* of a class of channel and linearly-constrained equaliser combinations. The prefilter notion was also used to show that the "optimum" linear constraint sequence is in fact the channel impulse response sequence. This idea forms the basis of a cost switching adaptive linear constraint strategy that we discuss for future work in Section 6.2.

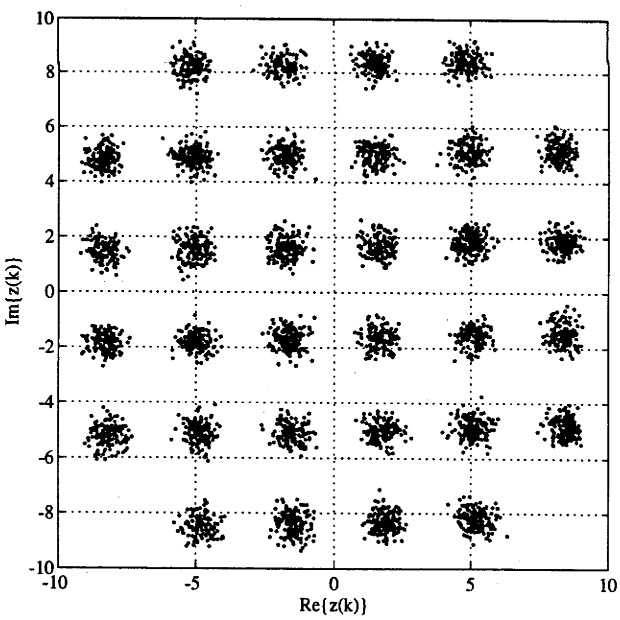
The issue of *finite equaliser parametrisation* was identified as an important one in light of recent reports on the inadmissibility of many of the popular blind equalisation algorithms under finite parametrisation due to a *local minima problem* [20, 21, 13]. However, our usage of *convex* cost functions has circumvented the problem because of their inherent ability to remain convex under finite parametrisation [24]. The convexity property also allows us to avoid the potentially difficult analysis of the algorithm in the finite parameter space that is typically necessary for algorithms that adopt *nonconvex* cost functions. Specifically, it is sufficient that we understand the *ideal* behaviour of the algorithm in the infinite parameter space with the confidence that this can be achieved with a degree of approximation that approaches the ideal as the number of parameters is made large.

We also investigated the case where the use of a severe level of *channel coding* may invalidate our assumption about the input symbol sequence being that all finite subsequences are probable. Our results indicted that the elimination of particular subsequences may lead to an alteration of the cost polytope whereby new vertices are introduced that may correspond to closed-eye equaliser parameter settings. As it is possible that the polytope touches the hyperplane at these vertices, minimisation of the cost may not necessarily be achieved by a zero-ISI equaliser parameter setting and therefore admissibility is lost. However, we placed this result in perspective by noting that this degenerate behaviour is only possible when *particular* subsequences are barred and that their lengths are less than or equal to the length of the t-space sequence corresponding to the cascade of the channel and equaliser. Also, manifestation of this degenerate behaviour depends on whether the constraint hyperplane is oriented accordingly. Furthermore, it is natural to expect degradation due to source correlation in *any* system identification procedure because it alters the persistence of excitation of that system.

To conclude this chapter, we presented an implementation of our algorithm and subsequently augmented our theoretical results via simulations.



(a)



(b)

Figure 5-9: Channel/Constraint 1 (generic convergence). 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$. a) Channel Output b) Equaliser output after 2000 iterations.

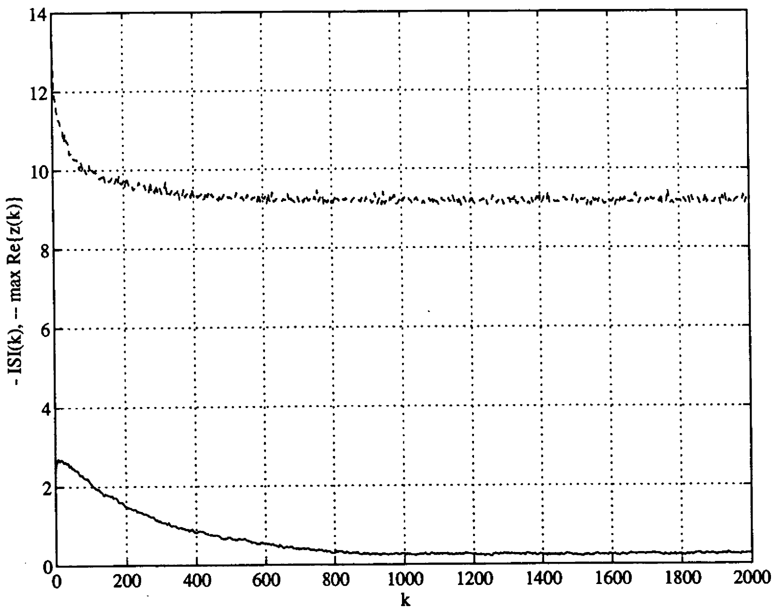


Figure 5-10: Channel/Constraint 1 (generic convergence). ISI and Cost evolution. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$.

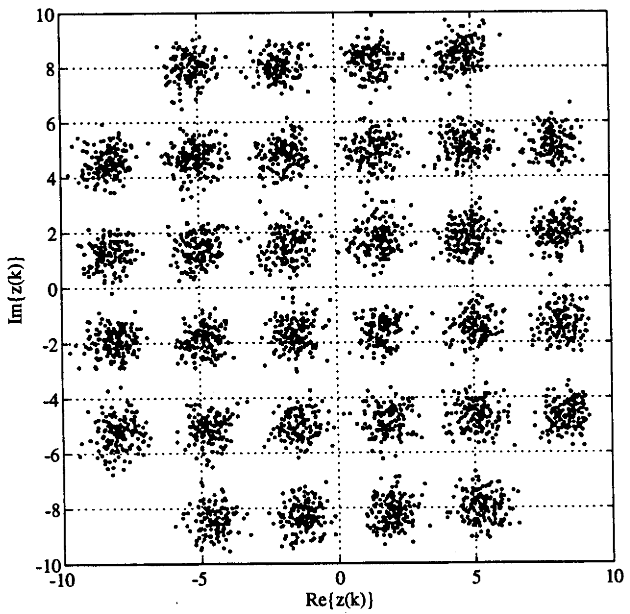


Figure 5-11: Channel/Constraint 1 (generic convergence). Equaliser output after 2000 iterations. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR = 30 dB.

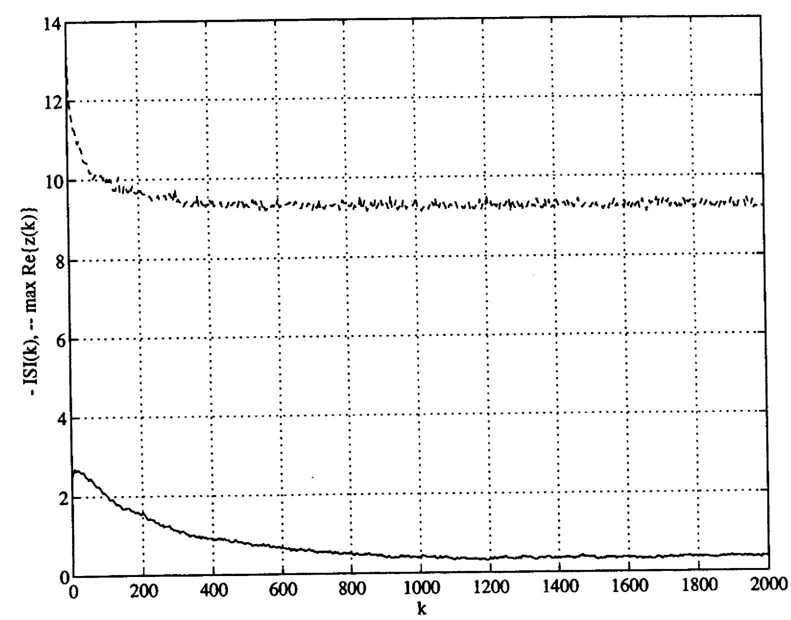


Figure 5-12: Channel/Constraint 1 (generic convergence). ISI and Cost evolution. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR $= 30$ dB.

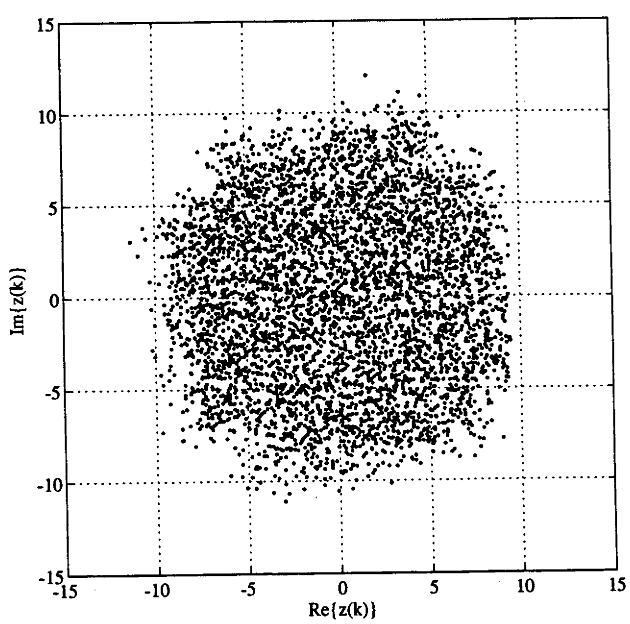


Figure 5-13: Channel/Constraint 1 (generic convergence). Equaliser output after 2000 iterations. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR $= 20$ dB.

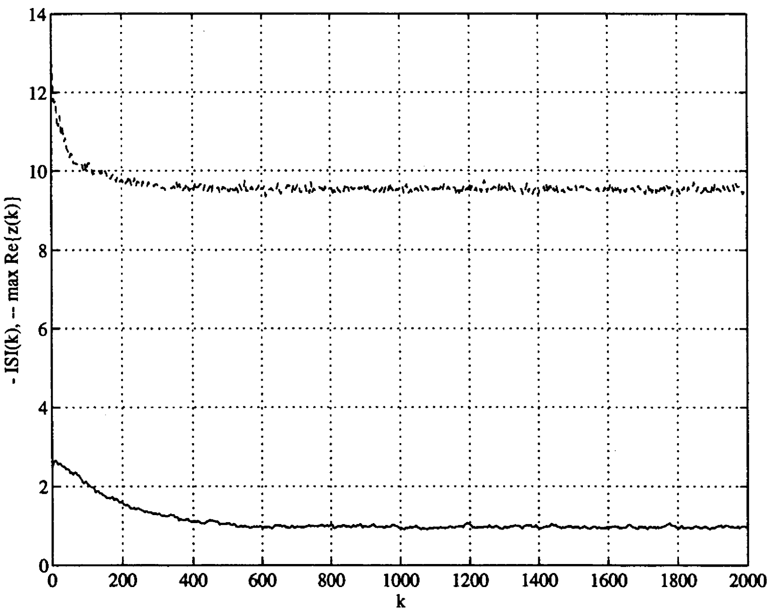


Figure 5-14: Channel/Constraint 1 (generic convergence). ISI and Cost evolution. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR = 20 dB.

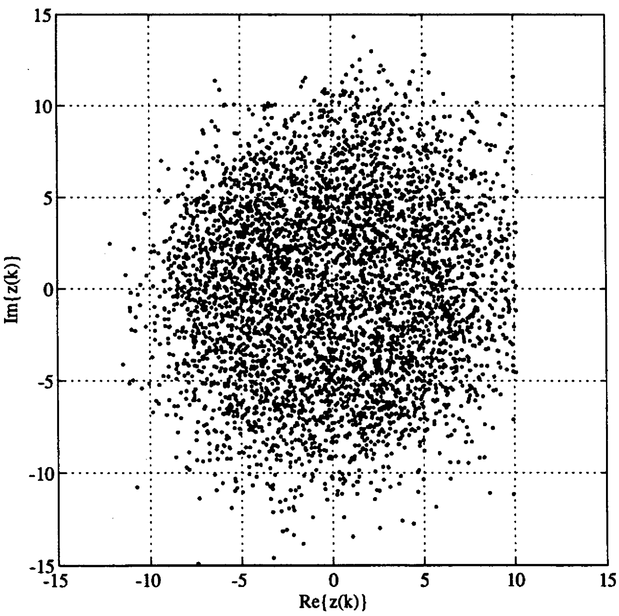


Figure 5-15: Channel/Constraint 1 (generic convergence). Equaliser output after 2000 iterations. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR = 10 dB.

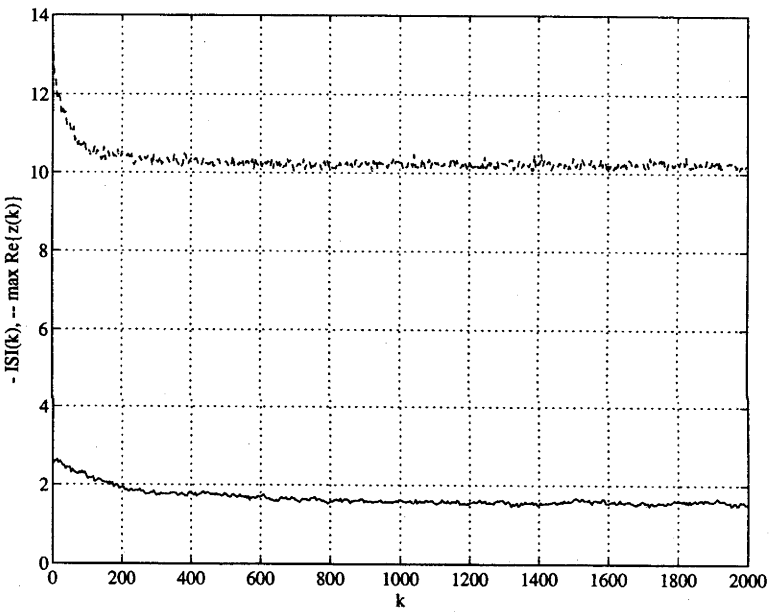
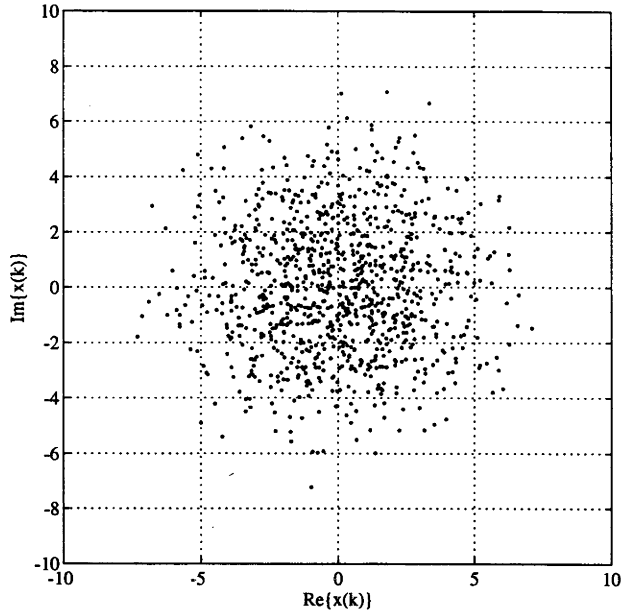
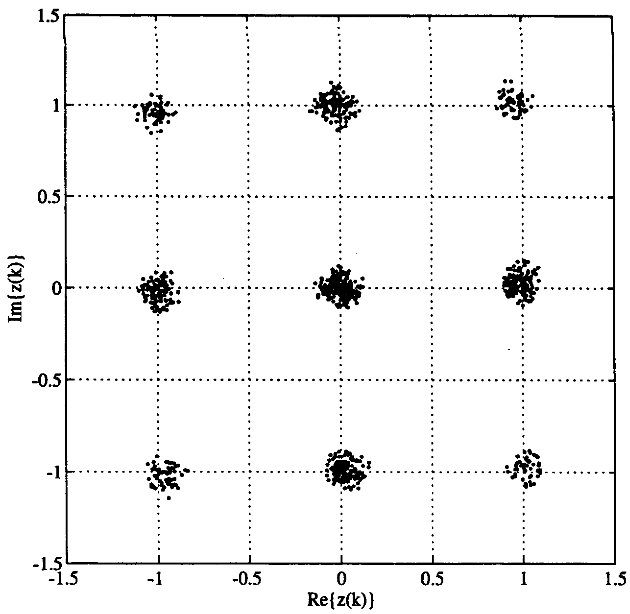


Figure 5-16: Channel/Constraint 1 (generic convergence). ISI and Cost evolution. 32-QAM, $L = 10$, $B = 5000$, $\mu = 0.001$, SNR $= 10$ dB.



(a)



(b)

Figure 5-17: Channel/Constraint 2 (nongeneric convergence). 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$. a) Channel Output b) Equaliser output after 4000 iterations.

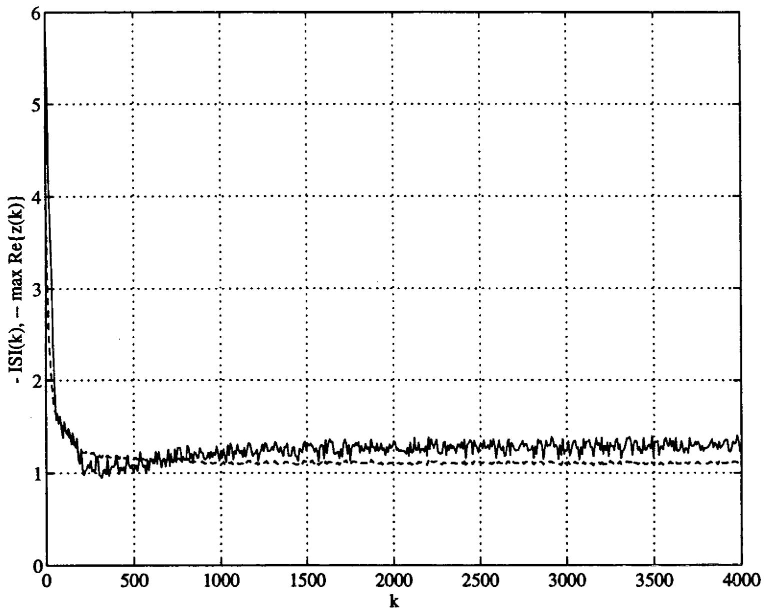


Figure 5-18: Channel/Constraint 2 (nongeneric convergence). ISI and cost evolution. 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$.

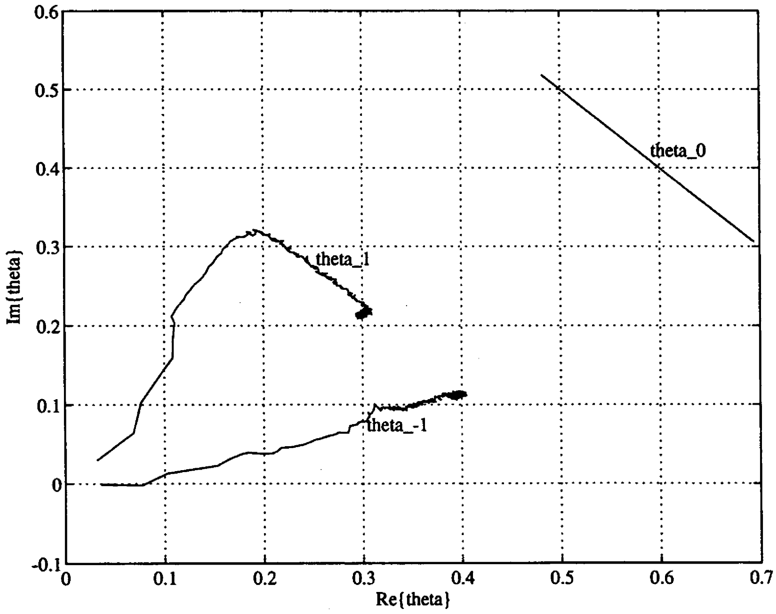


Figure 5-19: Channel/Constraint 2 (nongeneric convergence). Equaliser parameter trajectories. 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$.

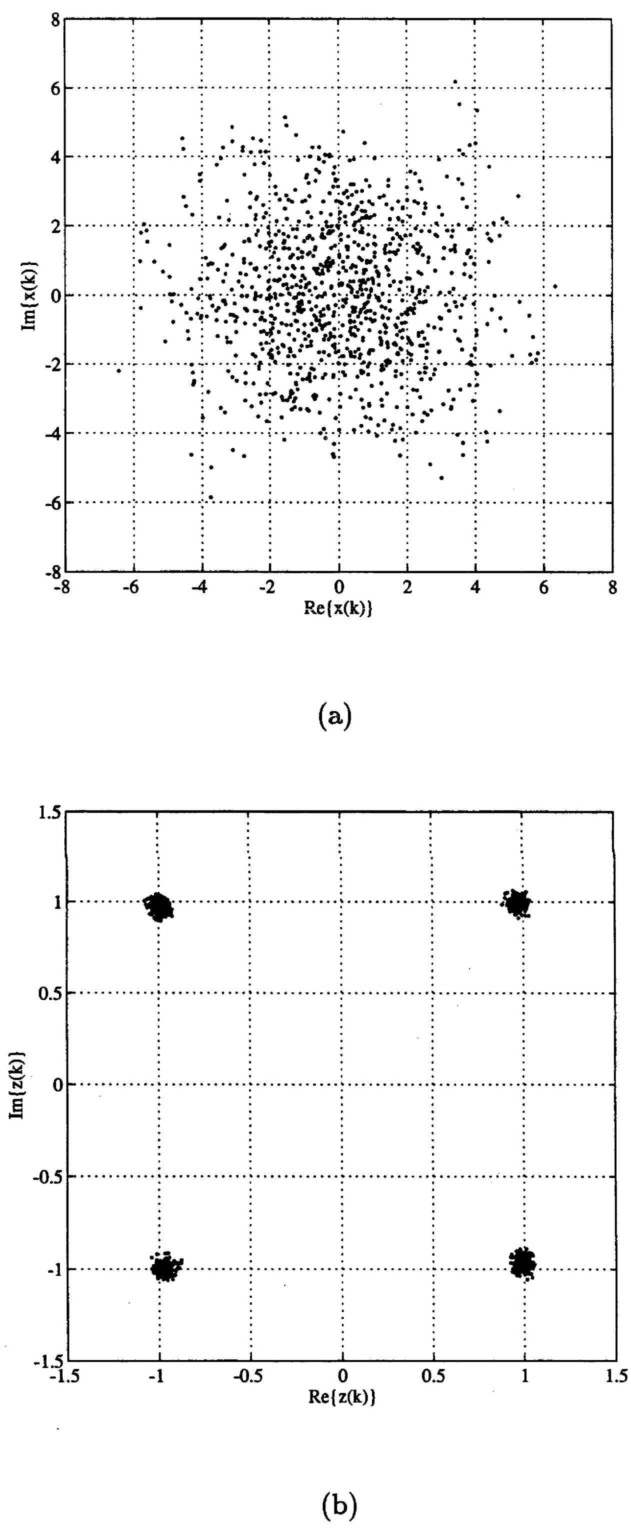


Figure 5-20: Channel/Constraint 3 (generic convergence after perturbing nongeneric channel). 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$. a) Channel Output b) Equaliser output after 10000 iterations.

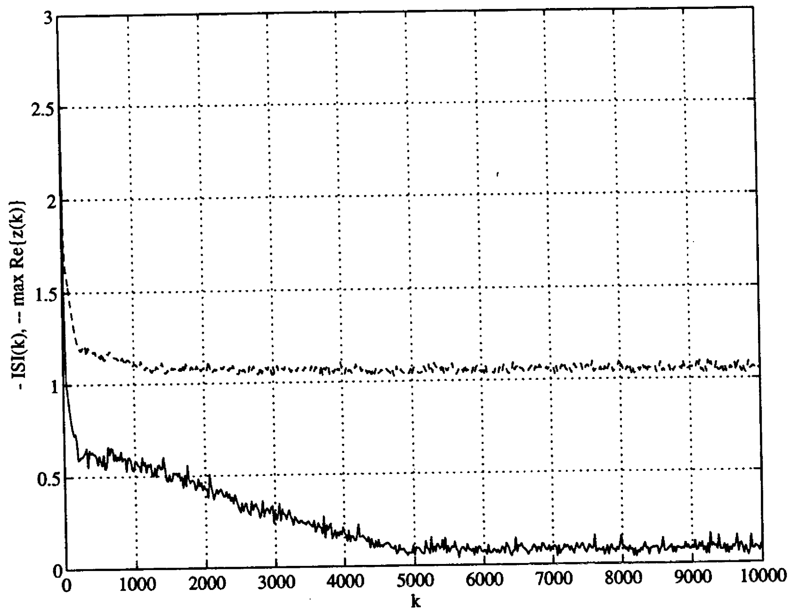


Figure 5-21: Channel/Constraint 3 (generic convergence after perturbing nongeneric channel). ISI and cost evolution. 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$.

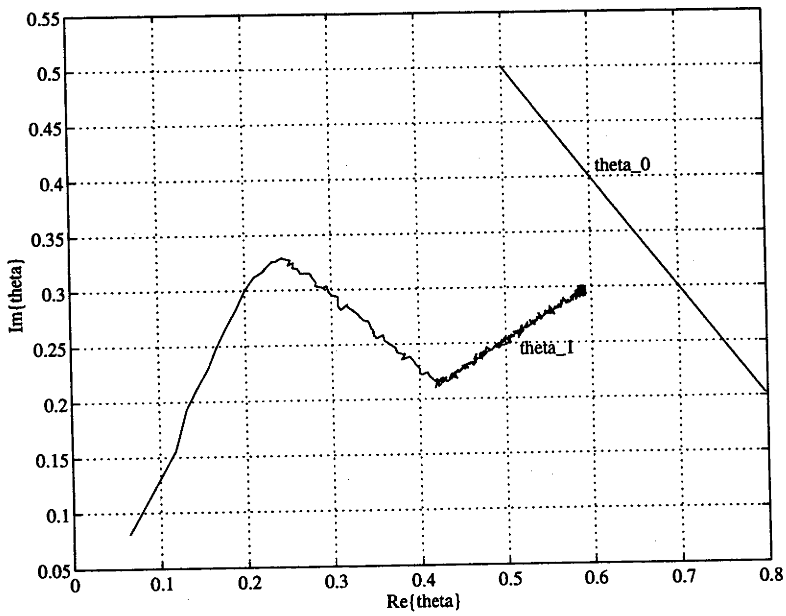


Figure 5-22: Channel/Constraint 3 (generic convergence after perturbing nongeneric channel). Equaliser parameter trajectories. 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$.

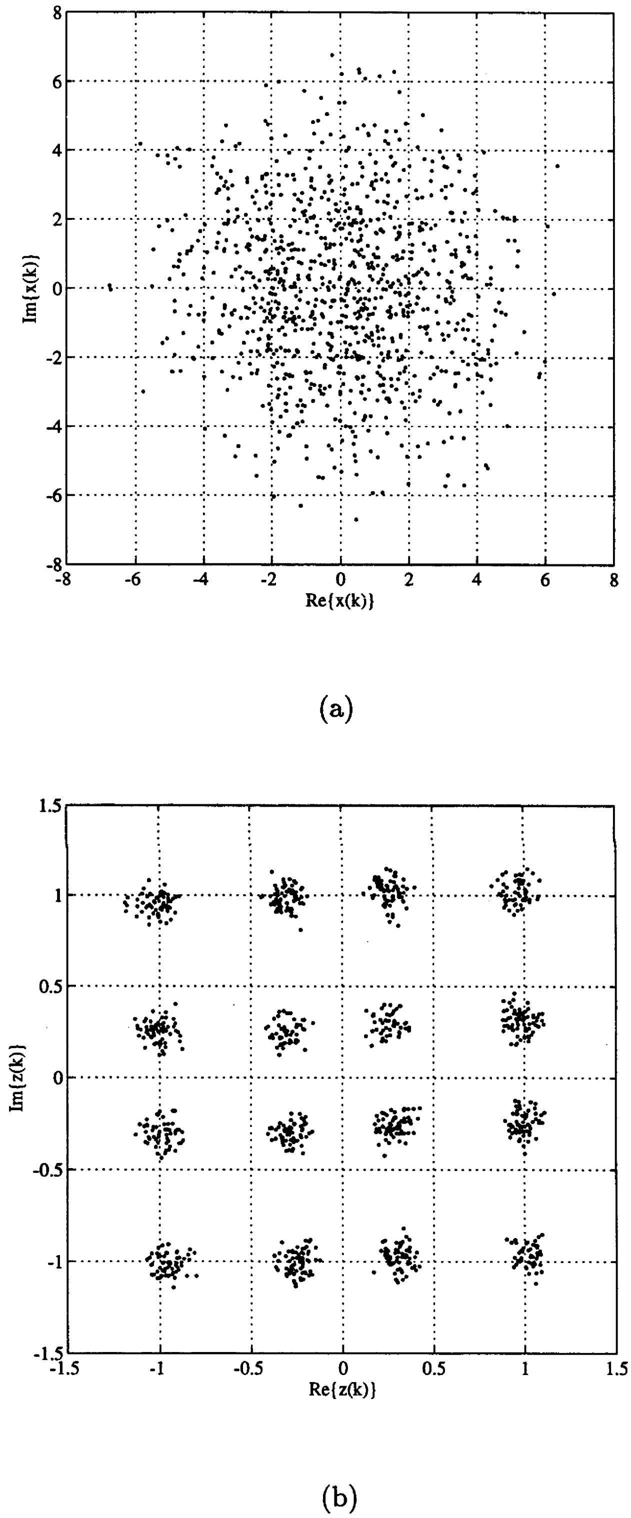


Figure 5-23: Channel/Constraint 4 (equivalent nongeneric channel/constraint). 4-QAM, $L = 21$, $B = 1000$, $\mu = 0.001$. a) Channel Output b) Equaliser output after 10000 iterations.

Chapter 6

Conclusions and Future Research

6.1 Conclusions

This thesis has been concerned with the development of a new approach to the design and analysis of a practical blind equalisation algorithm for QAM transmission systems by exploiting methods and results from *convex analysis.* The design aspect focused on developing an algorithm that is *admissible* under both *infinite* and *finite parametrisations* of the equaliser by considering cost functions that are *convex* in the equaliser parameters. To achieve such a convexity property it is sufficient to have the cost generated by a memoryless processing of the equaliser output using a scalar convex function.

We embraced a relatively new criterion of equalisation, namely, *equalisation without gain identification (EWGI)* [28, 29, 24, 26] which exploits a constraint on the equaliser parameters. Using this criterion, we developed a particular design philosophy to obtain a suitable convex cost. This, in effect, gave us an algorithm that is also *amenable to analysis* because the ideal behaviour of the algorithm can be characterised relatively easily in the infinite parameter space with the confidence that the algorithm will achieve arbitrarily close-to-ideal behaviour in the finite parameter space with a sufficient number of parameters. In contrast, it has been shown that other popular algorithms based on nonconvex cost functions do not possess this desirable feature as they exhibit ill-convergence under finite equaliser parametrisations [20, 21, 13].

On the analysis side, we have developed a novel *geometric* view of the cost minimisation problem that is sufficiently general to include all practical QAM constellations.

With it, we have proven the admissibility of our algorithm and have conveyed an intuitive understanding of the nongeneric mechanism of nonunique minimisation. We have justified our particular approach and have put it in its proper light by examining other approaches and identifying and understanding the conditions under which those algorithms fail. Although our algorithm, as we have implemented it, is perhaps not mature enough for mainstream application, it must be emphasised that the amenability of our algorithm to analysis opens the door to further extensions that may simplify its operation and/or enhance its performance. We now summarise the main ideas that have been developed in each chapter of the thesis.

In Chapter 1, we introduced the problem of blind equalisation as a specific case of *blind deconvolution*, that is, the general problem of effectively identifying the *inverse* of a linear system based solely on observations of its output and *a priori* knowledge of the statistics of the input. For blind equalisation, this identification must be done *on-line* and therefore there exists limits on the computational complexity of a blind equalisation algorithm if it is to be implemented in hardware. We provided a sampling of applications of blind equalisation and suggested that blind equalisation itself fulfills more naturally the idea of equalisation on the premise that equalisation should be entirely a receiver function. We also justified our focus on the *modified error signal algorithm (MESA)* class of algorithms [13] by extolling their relative computational simplicity in comparison to another class labelled *polyspectral* algorithms [17, 18] and highlighting the fact that a rigorous understanding of MESA's is far from mature despite their popularity.

We formulated the *blind equalisation problem* in detail in Chapter 2. There we gave our modelling assumptions on the input constellation, channel, and equaliser models and finally gave a formal statement of the blind equalisation problem. We also listed several practical considerations through which we forewarn of potential significant discrepancies between the ideal behaviour of an infinitely parametrised equaliser and its practical finitely parametrised counterpart.

Our justification of our new approach to blind equalisation begins in Chapter 3 where we collate the major contributions concerned with the design and analysis of MESA's from the literature and organise them into a proper classification that allows for the identification of approaches that offer benefits over the others. We described the MESA approach and stated a highly desirable criterion for such algorithms coined *admissibility* [22, 23]. We developed a suitable classification structure centered around the type of cost function, the degree of alphabet restoration implied, and whether the equaliser parametrisation is infinite or finite. The cost was either unimodal or multimodal where the former implies a global convergence behaviour and the latter implies local convergence. Two degrees of alphabet restoration were considered, that is, *alphabet restoration* and *equalisation without gain identification (EWGI)*. The former implies restoration of the symbol constellation at the output of the equaliser while the latter implies restoration up to an unknown complex gain factor (*i.e.*, scaling and rotation of the constellation).

The major results of the classification included the fact that most practical algorithms fit into the category of alphabet restoration algorithms that have multimodal costs. Although some of these algorithms were initially proposed and proven admissible under the convenient infinite equaliser parametrisation model, we cited some literature that established the inadmissibility of these algorithms under the practical finite parametrisation due to a *local minima problem* [20, 21, 13]. Also, the class of alphabet restoration algorithms with unimodal costs (a seemingly idealistic combination) was shown to be inadmissible. The combination of a EWGI algorithm with a unimodal cost was shown to possibly offer an admissible design due to its ability to distinguish between the different zero-ISI parametrisations that correspond to different delays of the equalised output. We showed that a fundamental characteristic of an EWGI algorithm is its imposition of a linear constraint on the equaliser parameters. It is this constraint that achieves this delay sensitivity at the expense of gain identification. However, we also demonstrated that this combination is susceptible to a *nonuniqueness phenomenon* where a particular (pathological) combination of the channel and constraint induces a particular dependence of the cost on the delay that is *not* unique. We argued that a fundamental consequence of this is that the cost may be minimised nonuniquely and by equaliser parameter settings that may not correspond to zero-ISI parameter settings. However, there always exists at least one zero-ISI parameter setting that minimises the cost and, in the case of nonuniqueness, at least two zero-ISI settings.

We also highlighted the advantages of using costs that are *convex* in the equaliser parameters and, in particular, cited the result that it is immune to the local minima problem owing to its ability to remain convex under finite parametrisation or any *linear* constraint for that matter [24]. A conclusion of this important chapter is that attention should be directed at the class of EWGI algorithms with convex costs under linear constraints on the equaliser parameters.

In Chapter 4 we documented the preliminary work on the design and analysis of a suitable EWGI algorithm using a convex cost with a linear constraint on the equaliser parameters. We presented the work as a series of nontrivial generalisations of the convex cost and simple parameter constraint proposed for real PAM systems by [29, 24, 26]. In the process, we introduced a *cost design philosophy* that followed from the EWGI approach and subsequently used it to justify the original work of [29, 24, 26] for real PAM systems and that of [34] for complex QAM systems. Our philosophy takes account of the issues relevant to the design of a suitable cost and constraint for QAM transmission such as the problem of the phase offset of the equaliser output. These issues supplement those that must be considered for real systems. We also provided a simple tool that permits a more straightforward analysis of the admissibility of an algorithm. The tool is the idea of expressing both the cost and constraint as functions of their associated *total parameter space* or t-space sequences. This simple idea allows the problem of linearly constrained cost minimisation to be viewed as a conventional optimisation problem. Viewing the problem in the t-space also has the advantage that an ideal zero-ISI equaliser parameter setting has the simple form of the Kronecker delta sequence. Despite its usefulness, the t-space formulation of the cost function assumes that the QAM constellation is *square* and that all subsequences of the symbol sequence are probable. Therefore, this analysis method can not be applied without modification to the case of nonsquare QAM constellations and in some channel coding scenarios where certain subsequences are prevented from occurring.

Our final generalisation of the cost and constraint of [29, 24, 26] is documented in Chapter 5 where we considered a convex cost subject to a general linear constraint on the equaliser parameters. We built upon the simple t-space analysis in Chapter 4 and established a more general framework in which the problem of constrained cost minimisation could be studied using a novel *geometric view* of the problem. The geometric view facilitates a more intuitive understanding of the problem in both the generic and nongeneric (nonuniqueness phenomenon) cases of cost minimisation. Moreover, unlike the results of the previous chapter, this framework can accommodate all practical QAM constellations. Obtaining the geometric view involves a translation of the problem into the t-space as before. However, this translation depends less on the particular form of the convex cost and QAM constellation used. Instead, two geometric objects corresponding to the cost and constraint were defined in the t-space, namely, a *cost level surface* and a *constraint hyperplane.* The orientation of the hyperplane was shown to depend on both the linear channel and the linear equaliser parameter constraint. The minimisation problem is then viewed as one of scaling the size of the cost level surface object until it "*touches*" the constraint hyperplane. The scaling factor at which touching occurs corresponds to the value of the cost and the intersection set determines the set of equaliser parameter settings that simultaneously conform to the constraint and minimise the cost. In this light, our most important result was to show that the cost level surface is the boundary of a *convex polytope* whose *vertices* correspond to zero-ISI (open-eye) equaliser parameter settings. Thus, the generic situation is one where a vertex touches the constraint hyperplane which implies global minimisation of the cost by a *particular* zero-ISI equaliser parameter setting. This was our *global convergence result.* To contrast, nonunique minimisation is viewed as the nongeneric situation where the hyperplane is oriented in such a way that the polytope touches the hyperplane on an *edge* or *face* (or higher dimensional analogue). This implies that all of the points (*i.e.*, zero-ISI equaliser parameter settings) that define the edge or face, *etc.*, minimise the cost which is clearly a case of nonunique minimisation. Such an edge or face, *etc.*, contains at least two vertices and hence at least two zero-ISI equaliser parameter settings. With this view, it became clear that the nonuniqueness phenomenon is attributed to a *particular* combination of the channel and general constraint because it is this combination that determines the particular degenerate orientation of the hyperplane. Using the insight gained, we were able to construct an example of nonunique minimisation and verified the result with simulations.

Another powerful tool that we developed was the *prefilter formulation* of a general linear constraint on the equaliser parameters. This formulation establishes the equivalence between an equaliser with a general linear constraint and the cascade of a *prefilter* and an equaliser with a single parameter constraint. Furthermore, the channel-equaliser-single tap constraint combination is shown to be a *canonical representation* of a class of channel and linearly-constrained equaliser combinations. The prefilter notion was also used to show that the "optimum" linear constraint sequence is in fact the channel sequence. It is optimum in the sense that it gives the greatest robustness against nonunique minimisation and is equivalent in the t-space to a configuration without ISI and an equaliser with a single-parameter constraint. This implies that, in fact, no equalisation is required or, equivalently, the unconstrained parameters of the equaliser would converge to zero during adaptation. This idea forms the basis of a switching cost adaptive linear constraint strategy that we discuss for future work in Section 6.2.

The issue of *finite equaliser parametrisation* was identified as an important one in light of recent reports on the inadmissibility of many of the popular blind equalisation algorithms under finite parametrisation due to a *local minima problem* [20, 21, 13]. However, our usage of *convex* cost functions has circumvented the problem because of the property that they maintain their convexity under finite parametrisation [24]. The convexity property can provide a guarantee that, under finite equaliser parametrisation, the minimum of the constrained cost function corresponds to an approximation of an ideal equaliser parameter setting that is arbitrarily close to the ideal one provided that the number of parameters is large enough (to any prescribed degree). This effectively allows us to avoid the potentially difficult analysis of the algorithm in the finite parameter space that is typically necessary for algorithms that adopt *nonconvex* cost functions.

We also investigated the case where the use of *channel coding* may invalidate our assumption about the input symbol sequence being that all finite subsequences are probable. Our results indicted that the elimination of particular subsequences may lead to an alteration of the cost polytope. We gave an example in which the altered polytope included vertices which did not correspond to zero-ISI equaliser parameter settings and subsequently demonstrated the ill-convergence of the equaliser parameters to one of these degenerate vertices. However we placed these result in perspective by noting that the ill-convergence depends on rather specific conditions and on the length of the omitted subsequences being relatively short compared to the channel and equaliser cascade. Furthermore, it is natural to expect degradation due to source correlation in *any* system identification procedure (blind or otherwise) because it reduces the necessary persistence of excitation of that system.

Finally, we presented an implementation of our algorithm and subsequently augmented our theoretical results via simulations.

6.2 Future Directions of Research

Whilst not an exhaustive list, we propose the following topics for future research that may serve to enhance and extend the research described in this thesis.

6.2.1 Dynamic Step-Size Normalisation

One distinguishing factor between the implementation of our algorithm in Chapter 5, Section 5.7 and the CMA algorithm is that the former requires a memory buffer with a size on the order of $2B$ where a typical value of B is 5000 (for 32-QAM). Although this does not appear to be too taxing given the addressing space of most DSP chips, there still may be justification for an implementation that is comparable to the CMA algorithm in terms of its usage of memory.

As the sole usage of the buffer is in the approximation of the $\max_{\{a_k\}}(\cdot)$ operation it is worthwhile to consider other methods to approximate this operation. For example, in [34], it is noted that $\max_{\{a_k\}} \mathrm{Re}\{z_k\} = \max_{\{a_k\}} |\mathrm{Re}\{z_k\}| = \|\mathrm{Re}\{z_k\}\|_\infty$. We may approximate the l_∞ norm by an l_p norm where p is large, specifically,

$$\|\mathrm{Re}\{z_k\}\|_\infty \approx \|\mathrm{Re}\{z_k\}\|_p = (\mathrm{E}[|\mathrm{Re}\{z_k\}|^p])^{\frac{1}{p}}, \quad p \text{ large.} \tag{6.2.1}$$

Ignoring the pth root operation, a suitable approximation of our cost is

$$\mathcal{J}(\theta) = \mathrm{E}[|\mathrm{Re}\{z_k\}|^p]. \tag{6.2.2}$$

A stochastic gradient descent of (6.2.2) involves using the instantaneous form of (6.2.2) and evaluating its gradient with respect to the equaliser parameters. The result is the following equaliser parameter update recursion.

$$\begin{aligned} \theta_i(k+1) &= \theta_i(k) - \mu \frac{\partial |\mathrm{Re}\{z_k\}|^p}{\partial \theta_i(k)} \\ &= \theta_i(k) - \mu |\mathrm{Re}\{z_k\}|^{p-2} \mathrm{Re}\{z_k\} \tilde{x}_{k-i} \quad \forall\, i. \end{aligned} \tag{6.2.3}$$

Note that the update quantity of (6.2.3) involves the computation of the $(p-2)$th power of the real part of the equaliser output. Since p is large, the step size factor μ will have to be exceedingly low to avoid the potential for numerical instability of the algorithm. However, a low value of μ implies an excessively slow convergence rate. One solution is to *normalise* the update term by some quantity that is guaranteed to be on the order of the high power term. If the normalisation factor is fixed then it must be initially chosen to be very high. However, this carries with it the consequence that the convergence rate of the algorithm will decrease substantially as the cost, and hence the equaliser output z_k, is reduced. Clearly a dynamic normalisation factor that depends on the ongoing size of the equaliser output is necessary in order to keep the convergence rate as high as possible. This is the approach taken in [34]. However, further work is still required to produce a normalisation scheme that represents a good balance between computational complexity and improved convergence rate.

Another aspect of our implementation that deserves attention is the implementation of our linear constraint. At present, a separate projection step is required after an unconstrained parameter update. This is not the most computational efficient method of implementing the constraint. It may be advantageous to implement a one-step linearly constrained adaptation procedure that avoids the separate projection step by using the General Sidelobe Canceller model described in [37].

6.2.2 Switching Cost Adaptive Linear Constraint Strategy

As alluded to in the discussion of the *prefilter formulation* of a linearly-constrained equaliser in Chapter 5, Section 5.5, a possible extension of the algorithm is the addition of a *switching cost and adaptive constraints strategy.* The intent of such as strategy would be to avoid nonuniqueness behaviour, but more important, to improve the convergence rate. The idea stems from the desirable situation that occurs when the constraint parameters are identical to their optimum values, namely, the channel impulse response values. The prefilter equivalence relation corresponding to this situation is given by (5.5.15) in Chapter 5 and is repeated below:

$$[\,h; g\bar{h}\,] \equiv [\,\delta; g\delta\,].$$

The left side of the relation denotes the combination of a channel with impulse response $\{h_i\}$ and an equaliser with parameters $\{\theta_i\}$ subject to a scaled channel sequence constraint

$$\mathrm{Re}\{\bar{g}h_0 * \theta_0\} = 1 \tag{6.2.4}$$

where the scale factor $g \in \mathbb{C}$ is arbitrary. The right side represents a completely equalised channel and an equaliser with parameters which we will denote by $\{\phi_i\}$. The zeroth parameter is constrained as follows:

$$\mathrm{Re}\{\bar{g}\delta_0 * \phi_0\} = \mathrm{Re}\{\bar{g}\phi_0\} = 1. \tag{6.2.5}$$

This relation indicates the equivalence between the combination of a channel and equaliser subject to a scaled channel sequence constraint and the combination of a completely equalised channel and an equaliser with a constraint on its zeroth parameter. The output of this equivalent equaliser is then

$$z_k = a_k * \theta_k = \sum_{i=-\infty}^{\infty} \phi_i a_{k-i}. \tag{6.2.6}$$

This makes the scaled channel sequence constraint, $\{\bar{g}h_i\}$, "*optimal*" in the sense that no equalisation is required. This suggests that all of the unconstrained equivalent equaliser parameters $\{\phi_i\}$ should effectively be adapted to zero. Although the parameters of the equivalent equaliser cannot be adapted directly the task can be accomplished by adapting the actual equaliser parameters $\{\theta_i\}$ to minimise a convex cost that does not necessarily have to be the *maximum complex deviation cost* described in Chapter 5. For example, if the input symbol sequence $\{a_k\}$ is uncorrelated and has zero mean, then minimising the energy of the output of the equaliser $\mathrm{E}\left[|z_k|^2\right]$ will suffice for effectively adapting the unconstrained equivalent equaliser parameters to zero. This is indicated by the following calculation:

$$\begin{aligned}
\mathcal{J}(\phi) &= \mathrm{E}\left[|z_k|^2\right] = \mathrm{E}\left[\left|\sum_{i=-\infty}^{\infty} \phi_i a_{k-i}\right|^2\right] \\
&= \sum_{i=-\infty}^{\infty} |\phi_i|^2 \mathrm{E}\left[|a_{k-i}|^2\right] \\
&= \mathrm{E}\left[|a_k|^2\right]\left[|\phi_0|^2 + \sum_{i\neq 0} |\phi_i|^2\right].
\end{aligned} \tag{6.2.7}$$

The squared modulus of the zeroth parameter $|\phi_0|^2$ may be evaluated for ϕ_0 satisfying the constraint of (6.2.5). The result is

$$\mathcal{J}(\phi) = \mathrm{E}\left[|a_k|^2\right]\left[\frac{1}{|g|^2} + \sum_{i\neq 0} |\phi_i|^2\right]. \tag{6.2.8}$$

The form of (6.2.8) clearly indicates that the minimum of $\mathcal{J}(\phi)$ is achieved when $\phi_i = 0,\ \forall\ i, i \neq 0$. Therefore, minimisation of the cost of (6.2.7) corresponds to zeroing all of the unconstrained equivalent equaliser parameters.

The problem with using the optimum constraint of (6.2.4) is that it assumes full knowledge of the channel and, hence, any algorithm which exploits this fact would be *nonblind*. We propose the following *switching cost and adaptive constraints strategy* which may avoid this limitation:

1. Initially adapting the parameters according to the original algorithm of Chapter 5, Section 5.1.

2. Then, periodically, on a longer time scale than the parameter adaptation, update the constraint with an *estimate* of $\{h_i\}$ formed from the inverse of the estimate of the channel inverse contained in the equaliser parameters $\{\theta_i\}$. This sets the constraint parameters to successive estimates of the optimum constraint parameters, *i.e.*, $\{h_i\}$. The longer time scale is needed to ensure the stability of the overall equaliser parameter adaptation. It allows the equaliser parameters to converge after each constraint update.

3. When the constraint has a sufficiently good approximation of $\{h_i\}$, then this corresponds to the situation where we have an approximately equalised channel and an equivalent equaliser (with a single fixed parameter) that need only zero its unconstrained parameters. This can be done by switching the cost to the energy cost (6.2.7) and adapting the actual equaliser parameters to minimise that cost.

Issues related to the extra computation required and the stability of the above two-time scale adaptive system must be addressed.

6.2.3 Convergence Test

In the blind context, establishing whether the equaliser has converged to an open-eye equaliser parameter setting is a difficult problem. Here we propose one method for the blind equalisation algorithm for real PAM systems proposed by [29, 24, 26] where the cost and constraint are

$$\mathcal{J}(\theta) = \max_{\{a_k\}} |z_k| \quad \text{subject to } \theta_0 = 1. \tag{6.2.9}$$

Our method exploits the unique relationship between the l_1 and l_2 norms of the t-space sequences corresponding to zero-ISI equaliser parameter settings. The l_1 norm $\|\cdot\|_1$ and l_2 norm $\|\cdot\|_2$ of an arbitrary t-space sequence are defined by

$$\|t\|_1 \triangleq \sum_{i=-\infty}^{\infty} |t_i|, \tag{6.2.10}$$

$$\|t\|_2 \triangleq \left(\sum_{i=-\infty}^{\infty} t_i^2\right)^{\frac{1}{2}}. \tag{6.2.11}$$

Figure 6-1 depicts the relationship between the l_1 and l_2 norms by giving the loci of equal l_1 and l_2 norms in the t-space in the case where the t-space is two-dimensional.

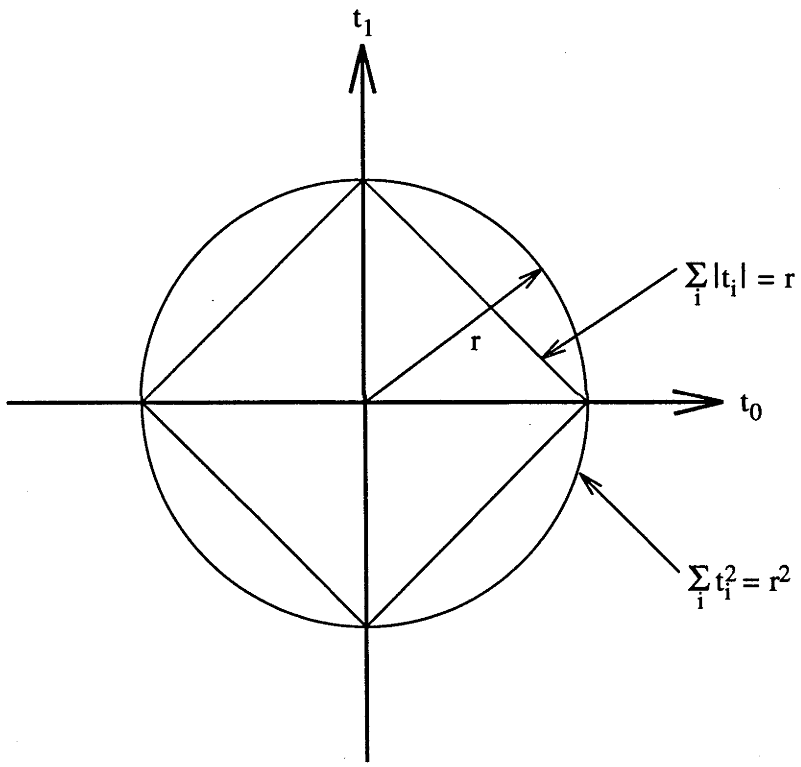


Figure 6-1: Loci of equal l_1 and l_2 norms of $\{t_i\}$ in the 2-D case.

Note that loci only intersect at the vertices of the l_1 locus and that the vertices correspond to zero-ISI equaliser parameter settings. This result can be expected to hold for higher dimensions. Convergence to a zero-ISI equaliser parameter setting is then tested in the t-space by checking whether the l_1 and l_2 norms of the corresponding t-space sequence are identical. Note that this test is nonblind in the sense that the knowledge of the corresponding t-space sequence implies knowledge of the channel impulse response.

However, if we make the assumption that the symbols $\{a_k\}$ of the PAM constellation $\mathcal{A}$ are *uncorrelated* and have *zero mean*, that is,

$$\mathrm{E}\left[a_{k-i}a_{k-r}\right] = \begin{cases} \mathrm{E}\left[a_k^2\right], & i = r \\ 0, & i \neq r \end{cases} \tag{6.2.12}$$

then it is possible to observe both norms *implicitly* at the output of the equaliser as now revealed. Expressing the equaliser output $\{z_k\}$ in terms of the t-space parameters we have

$$z_k = \sum_{i=-\infty}^{\infty} t_i a_{k-i}, \quad \forall\, k. \tag{6.2.13}$$

The norm $\|t\|_2$ can be implicitly obtained from the *average energy* of the equaliser output by

$$\begin{aligned} \mathrm{E}\left[z_k^2\right] &= \mathrm{E}\left[\sum_{i=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} t_i t_r a_{k-i} a_{k-r}\right] \\ &= \sum_{i=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} t_i t_r \mathrm{E}\left[a_{k-i}a_{k-r}\right] \\ &= \mathrm{E}\left[a_k^2\right] \sum_{i=-\infty}^{\infty} t_i^2. \end{aligned} \tag{6.2.14}$$

where we have utilised our assumption that the symbols are uncorrelated and have zero mean. Assuming the symbol sequence is stationary, then the average energy of the symbols $\mathrm{E}\left[a_k^2\right] = \mathcal{E}_a$ is constant. Note that the average energy of an equaliser output value, $\mathrm{E}\left[z_k^2\right]$, is a scaled l_2 norm of the t-space sequence.

Provided Assumption 2.1 holds the norm $\|t\|_1$ is implicitly obtained by the equivalent expression of the cost function (6.2.9) in terms of the corresponding t-space sequence:

$$\max_{\{a_k\}} |z_k| = M \sum_{i=\infty}^{\infty} |t_i| \tag{6.2.15}$$

where $M = \max_{a \in \mathcal{A}} |a|$. Note that the cost is a scaled l_1 norm of the t-space sequence.

Solving for $\|t\|_1$ and $\|t\|_2$ in (6.2.15) and (6.2.14) and equating the results, we obtain the following condition that exists when the t-space sequence corresponds to that of a zero-ISI equaliser parameter setting.

$$\frac{\max_{\{a_k\}} |z_k|}{M} = \left(\frac{\mathrm{E}\left[z_k^2\right]}{\mathrm{E}\left[a_k^2\right]}\right)^{\frac{1}{2}} \tag{6.2.16}$$

Therefore, a practical test of convergence to a zero-ISI equaliser setting may be

$$\left| \max_{\{a_k\}} |z_k|_{\mathrm{est}} - M \left(\frac{\mathrm{E}\left[z_k^2\right]_{\mathrm{est}}}{\mathcal{E}_a}\right)^{\frac{1}{2}} \right| < \epsilon \tag{6.2.17}$$

where an estimate of the cost $\max_{\{a_k\}} |z_k|_{\text{est}}$ and estimate of the average energy of the equaliser output $\mathrm{E}\,[z_k^2]_{\text{est}}$ are used and ϵ is a tolerance factor.

Further generalisation of this result for use with our blind equalisation algorithm for complex QAM systems in Chapter 5 is desirable.

6.2.4 Nonlinearly-Constrained Adaptation

Here we propose the use of a simple *nonlinear* equaliser parameter constraint to deal with the case of *nonunique minimisation* for the algorithm proposed for real PAM systems by [29, 24, 26]. The associated cost and constraint is given by (6.2.9).

Consider the geometric view of the cost minimisation situation in the t-space depicted in Figure 6-2 for the case where the t-space is two-dimensional and uncorrelated, zero mean, binary PAM symbols are assumed.

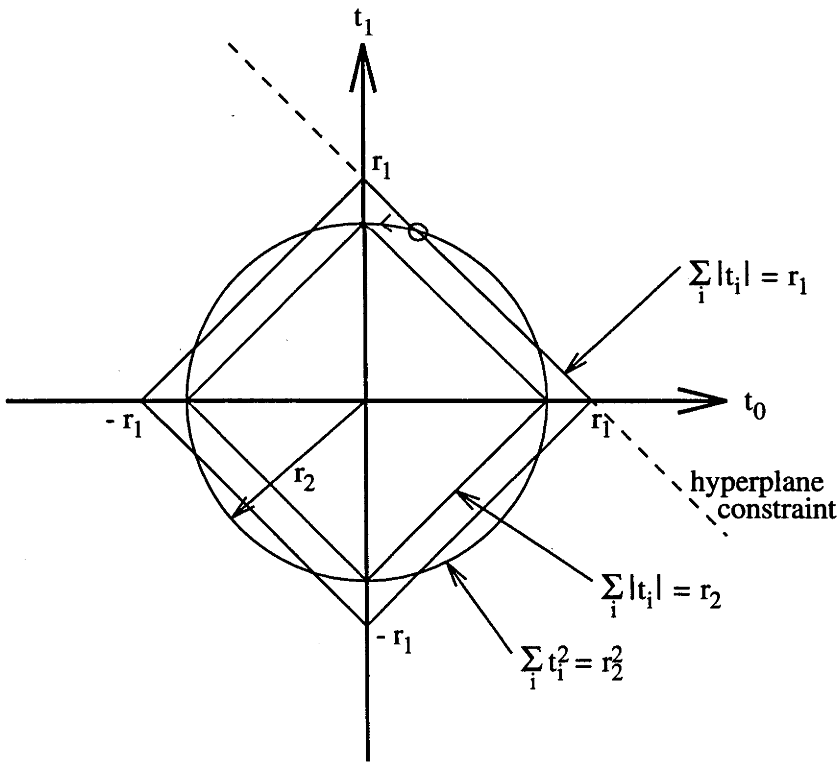


Figure 6-2: Geometric view of a nonunique equaliser parameter setting denoted by 'o'. A particular energy (circular) constraint is imposed to force the parameter adaptation (in the direction indicated) to a zero-ISI equaliser parameter setting corresponding to a lower cost ($r_2 < r_1$).

Due to the assumptions on the binary symbols, the equivalent representation of the cost is given by (6.2.15) with $M = 1$. Note also that the circle represents the locus of equal average equaliser output energy as given by (6.2.14) with $\mathrm{E}[a_k^2] = 1$. The relevance of the equaliser output energy will be discussed later.

Observe the hypothetical situation in Figure 6-2 where the *constraint hyperplane* touches the *cost polytope* corresponding to a cost of r_1 on an *edge*. Therefore, a nonunique minimisation situation exists. Any equaliser parameter setting corresponding to a t-space sequence that lies in the intersection set (a line in this case) between the cost polytope and the hyperplane achieves a minimum cost value of r_1. However, only the *vertices* of the intersection set (the end points of the line in this case) correspond to zero-ISI equaliser parameter settings since they are also the vertices of the cost polytope.

Suppose that, as a result of the gradient descent based adaptation, the equaliser parameters are adapted to a cost-minimising setting that corresponds in the t-space to the point denoted by 'o' in Figure 6-2. This unstable point is not a zero-ISI equaliser parameter setting. The method we outline here to resolve this problem is to apply *locally* a nonlinear constraint that contains this point and can force the adaptation of the equaliser parameters to one of the zero-ISI settings at the vertices of the intersection set. The nonlinear constraint we propose is an *equaliser output energy constraint* represented by the circle in Figure 6-2. In a higher dimensional t-space, it would be a *hypersphere*. Note that the circle defines the locus of points of equal equaliser output energy where the energy is the energy resulting from an equaliser parameter setting corresponding to the point 'o' in the t-space. However, the corresponding values of the cost function are *not* the same. The equaliser parameter adaptation under this energy constraint corresponds to an adaptation of the equivalent t-space parameters from the point 'o' to the vertex (corresponding to a zero-ISI equaliser parameter setting) indicated by the arrow. This will occur because the cost evaluated along the nonlinear constraint in the direction of the arrow is monotonically decreasing. The cost evaluated in the other direction is locally increasing. Therefore, at least intuitively, we have justified the use of the energy constraint in dealing with the case of nonunique minimisation.

The following is a sketch of the procedure to implement our local nonlinearly constrained equaliser parameter adaptation.

1. Linearly-constrained equaliser parameter update phase until the cost (6.2.9) is

minimised [29, 24, 26].

2. Let $\mathcal{E} = \mathrm{E}\,[z_k^2]_{\mathrm{est}}$ define an estimate of the current average energy of the equaliser output.

3. Nonlinearly-constrained equaliser parameter update phase

   (a) normal gradient descent parameter update

   (b) normalise parameters by σ to maintain $\mathrm{E}\,[z_k^2]_{\mathrm{est}} \doteq \mathcal{E}$.

$$\theta_i' = \sigma\theta_i, \quad \forall i \tag{6.2.18}$$

   where

$$\sigma = \left(\frac{\mathcal{E}}{\mathrm{E}\,[z_k^2]_{\mathrm{est}}}\right)^{\frac{1}{2}} \tag{6.2.19}$$

4. Continue nonlinearly-constrained equaliser parameter update phase until cost re-converges.

The nonlinearly constrained equaliser parameter adaptation scheme was tested by running the following simulations. The channel used is defined in Figure 6-3. It is purposely constructed so that the impulse response of its inverse possesses two identical peaks in magnitude which causes nonunique minimisation behaviour of the algorithm of [29, 24, 26]. The geometric view of this condition is similar to that given by Figure 6-2. Figure 6-4 gives the equaliser output after 50000 iterations of the algorithm of [29, 24, 26] on binary PAM symbols without the nonlinearly constrained adaptation phase. Figure 6-5 gives the corresponding ISI and cost evolution. Note that the nonunique minimisation has resulted in the adaptation of the equaliser parameters to a closed-eye equaliser parameter setting. The algorithm was then run with the nonlinearly constrained equaliser parameter adaptation phase switched on after the cost converged (after 15000 iterations). Figure 6-6 shows the equaliser output after 100000 iterations of the algorithm with the nonlinear parameter adaptation switched on after 15000 iterations. Figure 6-7 shows the corresponding ISI and cost evolution. These figures indicate that an open-eye condition has been achieved at the output of the equaliser as a result of the nonlinear parameter adaptation phase.

Remarks

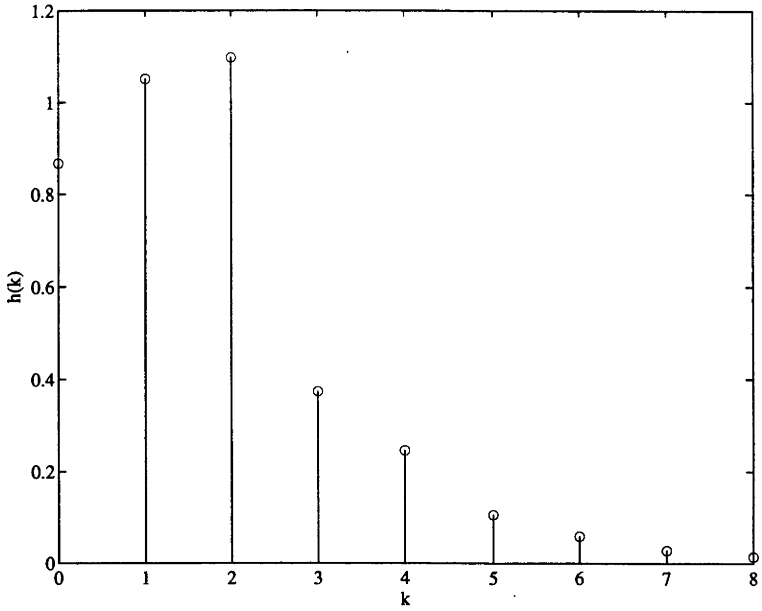
1. We emphasise that the nonlinear constraint is applied as a *local* constraint, that is, at an equaliser parameter setting that minimises the cost. The nonlinearly constrained convergence result may not apply assuming an initially arbitrary equaliser parameter setting as the constraint is nonlinear and thus may destroy the (global) convexity of the cost function.

2. Figure 6-7 illustrates an initially slow convergence of the ISI and cost after the nonlinear constraint is invoked. This seems to suggest that the equaliser parameters were at an unstable equilibrium point. It is possible that this situation arises when the initially closed-eye setting at 'o' of Figure 6-2 occurs precisely in the middle of the line of intersection. An energy constraint invoked at that point makes that point an unstable equilibrium point because the cost is lower in either directions along the constraint. A kickstart (momentary burst in the adaptation step-size) may serve to knock the adaptation of the equaliser parameters out of the unstable equilibrium and thereby subdue the initially slow convergence characteristic. A similar effect may be obtained by adding a small dithering (random) signal to the parameter update quantity.

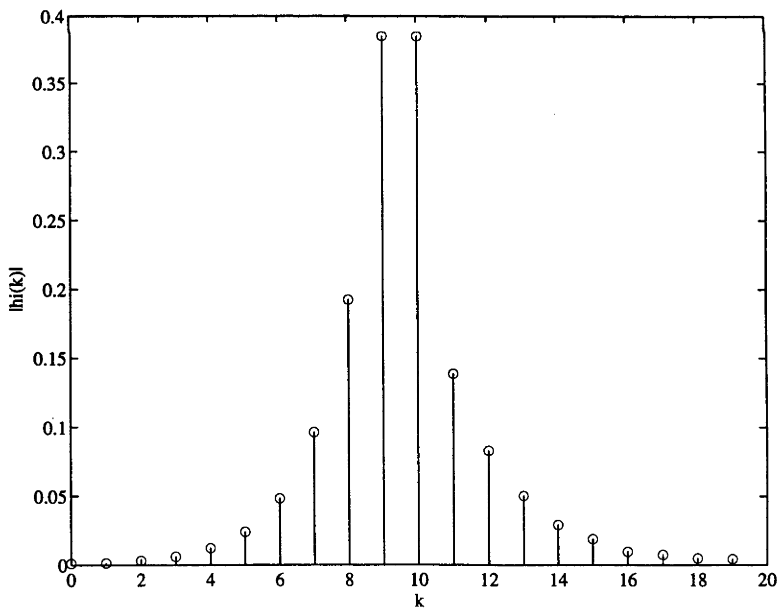
3. Further generalisation of this nonlinear constraint for use with the blind equalisation algorithm for complex QAM systems is desirable.

6.3 Final Word

In this thesis, we focused on a highly desirable property of blind equalisation algorithms known as *admissibility* and, subsequently, proposed one algorithm for QAM transmission systems based on a linearly-constrained convex cost that is *generically admissible*. That is, our algorithm is admissible except for particular (pathological) combinations of the linear constraint and channel. A similar algorithm was proposed for real PAM systems [29, 24, 26] and was also shown to be generically admissible in the same sense. However, to the knowledge of the author, there are currently no blind equalisation algorithms that can claim the distinction of admissibility for *all* practical channels. Therefore, despite the inception of the first blind equalisation algorithm in 1975 by Sato, the goal of a truly admissible algorithm has remained elusive for almost two decades! Fingers are indeed crossed in hope that progress will continue to this end and that such an algorithm, perhaps, may constitute the work of a future thesis on blind equalisation.



a)



b)

Figure 6-3: a) Impulse response of channel that causes nonunique minimisation behaviour. b) Magnitude of impulse response of channel inverse.

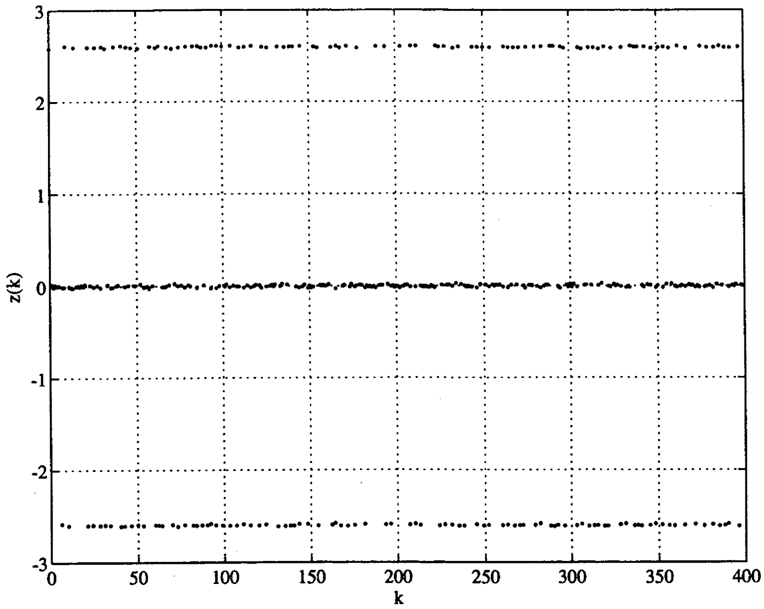


Figure 6-4: Equaliser output after 50000 iterations. Channel of Figure 6-3, binary PAM, $L = 21$, $B = 500$, $\mu = 0.0001$. A window of 400 output samples is shown.

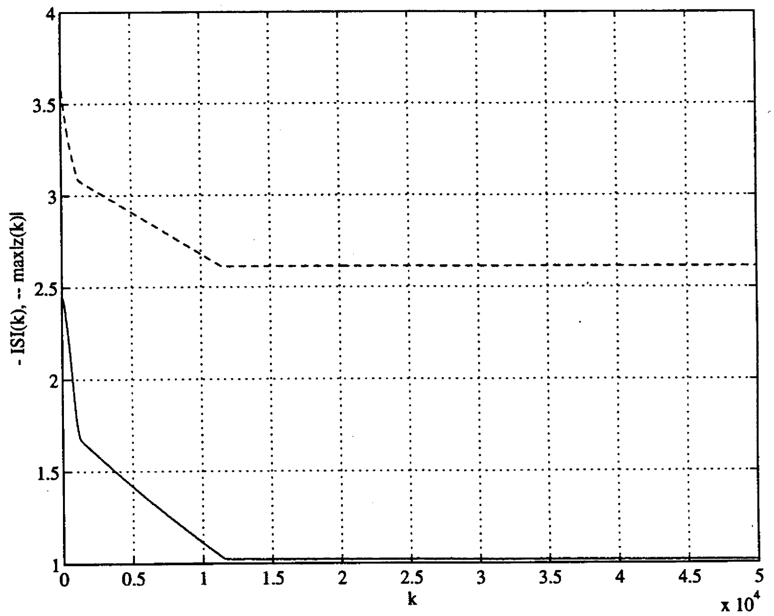


Figure 6-5: ISI and cost evolution. Channel of Figure 6-3, binary PAM, $L = 21$, $B = 500$, $\mu = 0.0001$.

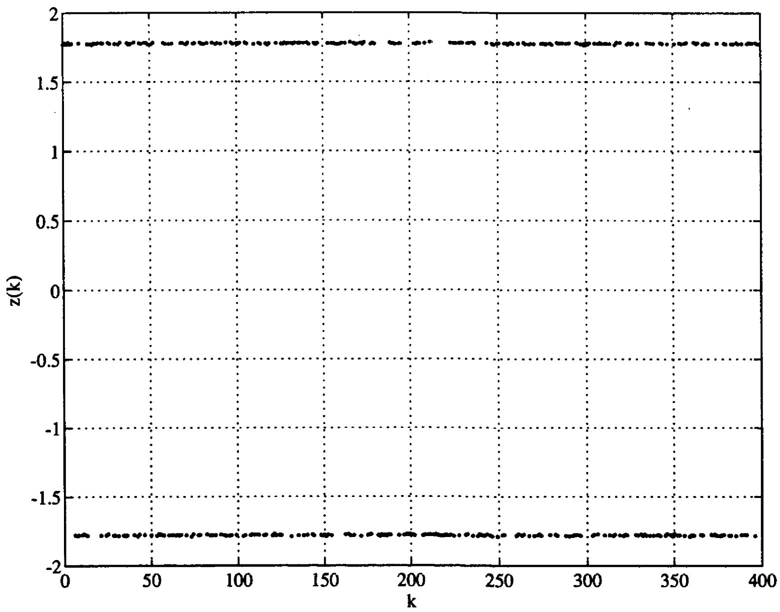


Figure 6-6: Equaliser output after 100000 iterations. Nonlinear constraint is invoked after 15000 iterations. Channel of Figure 6-3, binary PAM, $L = 21$, $B = 500$, $\mu = 0.0001$. A window of 400 output samples is shown.

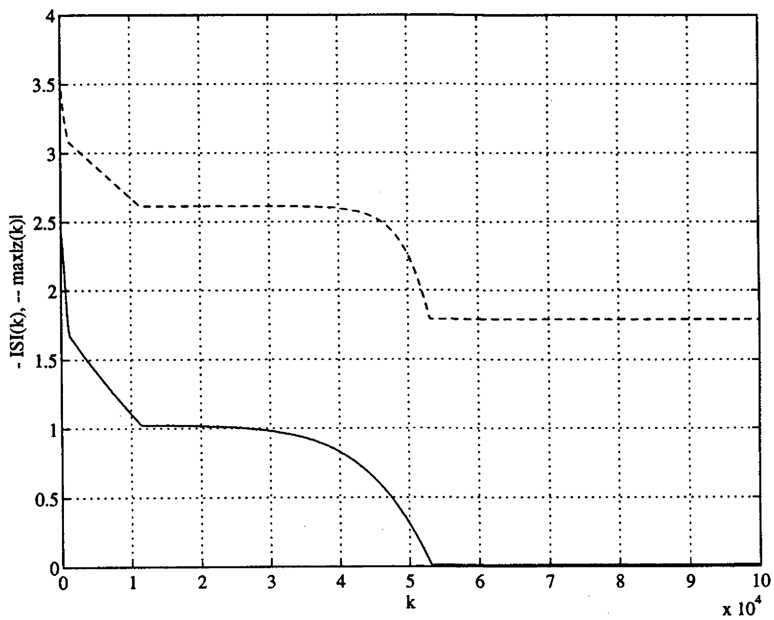


Figure 6-7: ISI and cost evolution. Nonlinear constraint is invoked after 15000 iterations. Channel of Figure 6-3, binary PAM, $L = 21$, $B = 500$, $\mu = 0.0001$.

Bibliography

[1] A.V. Oppenheim and R.W. Schafer. *Digital Signal Processing.* Prentice-Hall, Inc., New Jersey, 1975.

[2] R.A. Wiggins. Minimum entropy deconvolution. *Geoexploration*, 16:21–35, February 1978.

[3] R. Godfrey and F. Rocca. Zero memory non-linear deconvolution. *Geophysical Prospecting*, 29:189–228, 1981.

[4] F.K. Li, D. Held, J. Curlander, and C. Wu. Doppler parameter estimation for spaceborne synthetic aperature radars. *IEEE Trans. on Geoscience and Remote Sensing*, 23:47–51, January 1985.

[5] J.L. Yen. Image reconstruction in synthesis radiotelescope arrays. *Array Signal Processing*, pages 293–350, 1985. S. Haykin, Ed.

[6] Y. Sato. A method of self-recovering equalization for multi-level amplitude modulation. *IEEE Trans. on Communications*, COM-23:679–682, June 1975.

[7] D. N. Godard. Self-recovering equalization and carrier tracking in two-dimensional data communication systems. *IEEE Trans. on Communications*, COM-28:1867–1875, November 1980.

[8] N. K Jablon. Joint blind equalization, carrier recovery, and timing recovery for high-order QAM signal constellations. *IEEE Trans. on Signal Processing*, 40:1383–1398, June 1992.

[9] F.J. Ross and D.P. Taylor. An enhancement to blind equalization algorithms. *IEEE Trans. on Communications*, 39(5):636–639, May 1991.

[10] L. Tong, G. Xu, and T. Kailath. Blind identification and equalization of multipath channels. In *IEEE International Conference on Communications ICC-92*, Chicago, U.S.A., June 1992.

[11] J. R. Treichler and M. G. Agee. A new approach to multipath correction of constant modulus signals. *IEEE Trans. on Acoustics, Speech, and Signal Processing*, ASSP-31:349–472, April 1983.

[12] J.R. Treichler and M.G. Larimore. New processing techniques based on the constant modulus adaptive algorithm. *IEEE Trans. on Acoustics, Speech, and Signal Processing*, ASSP-33:420–431, April 1985.

[13] Z. Ding. *Application aspects of blind adaptive equalizers in QAM data communications.* PhD thesis, Cornell University, August 1990.

[14] C. L. Nikias. Arma bispectrum approach to nonminimum phase system identification. *IEEE Trans. on Acoustics, Speech, and Signal Processing*, 36:513–525, April 1988.

[15] B. Friedlander and B. Porat. Asymptotically optimal estimation of ma and arma parameters of non-gaussian process from high-order statistics. *IEEE Trans. on Automatic Control*, 35:27–36, January 1990.

[16] J. K. Tugnait. Identification of a non-minimum phase linear stochastic systems. *Automatica*, 22(4):457–464, 1986.

[17] D. Hatzinakos and C. L. Nikias. Blind equalization using a tricepstrum based algorithm. *IEEE Trans. on Communications*, 1990. under review.

[18] B. Friedlander and B. Porat. Adaptive channel equalization based on high-order statistics. In *Proceedings ICASSP-89*, pages 2278–2282, Glasgow, Scotland, May 1989.

[19] A. Benveniste, M. Goursat, and G. Ruget. Robust identification of a nonminimum phase system: blind adjustment of a linear equalizer in data communications. *IEEE Trans. on Automatic Control*, AC-25:385–399, June 1980.

[20] R. A. Kennedy, B. D. O. Anderson, Z. Ding, and C. R. Johnson, Jr. On the failure of proposed blind recursive identification schemes based on the Sato algorithm and its generalizations. In *International Symposium on Signal Processing and its Applications*, pages 267–270, Gold Coast, Australia, August 1990.

[21] Z. Ding, R. A. Kennedy, B. D. O. Anderson, and C. R. Johnson, Jr. Ill-convergence of Godard blind equalizers in data communications. *IEEE Trans. on Communications*, pages 1313–1327, September 1991.

[22] Z. Ding, C.R. Johnson, and R.A. Kennedy. On the admissibility of blind adaptive equalizers. In *IEEE International Conference on Acoustics, Speech, and Signal Processing ICASSP-90*, pages 1707–1710, 1990.

[23] R.A. Kennedy, K. Yamazaki, Z. Ding, S. Verdú, and S. Vembu. Approaches to blind equalization: Open problems. In *4th IFAC Intl. Symp. on Adaptive Systems in Control and Signal Processing ACASP-92*, pages 627–632, Grenoble, France, July 1992.

[24] S. Vembu, S. Verdú, R.A. Kennedy, and W.A. Sethares. Convex cost functions in blind equalization. In *Conference on Information Sciences and Systems*, pages 792–797, Baltimore, MD, March 1991.

[25] K. Yamazaki and R.A. Kennedy. Reformulation of linearly-constrained adaptation and its application to blind equalization. *IEEE Trans. on Signal Processing*, June 1994. to appear.

[26] W.A. Sethares, R.A. Kennedy, and Zhenguo Gu. An approach to blind equalization of non-minimum phase systems. In *IEEE International Conference on Acoustics, Speech, and Signal Processing ICASSP-91*, pages 1529–1532, Toronto, Canada, 1991.

[27] Z. Ding. Joint blind equalization and carrier recovery of QAM systems. In *Conference on Information Sciences and Systems*, pages 786–791, Baltimore, MD, March 1991.

[28] W.T. Rupprecht. Analysis of the mean-absolute-deviation-error for adaptive equalizers in data transmission systems. In *Proc. IASTED Int. Symp. Applied Signal Processing and Signal Filtering*, pages 99–102, Paris, 1985.

[29] S. Verdú, B.D.O. Anderson, and R.A. Kennedy. Anchored blind equalization. In *Proc. 25th Conf. on Info. Sci., and Systems*, pages 774–779, Baltimore, MD, March 1991.

[30] R. A. Kennedy, B. D. O. Anderson, Z. Ding, and C. R. Johnson, Jr. Local stable minima of the Sato recursive identification scheme. In *29th International Conference on Decision and Control*, Honolulu, HI, 1990. accepted.

[31] Z. Ding, C.R. Johnson, Jr., and R.A. Kennedy. On the (non)existence of undesirable equilibria of Godard blind equalizers. *IEEE Trans. on Signal Processing*, 1991. to appear.

[32] P. Bloomfield and W.L. Steiger. *Least Absolute Deviations.* Birkhäuser, Boston, MA, 1983.

[33] Z. Ding. *Application aspects of blind adaptive equalizers in QAM data communications.* PhD thesis, Cornell University, August 1990.

[34] R.A. Kennedy and Z. Ding. Blind adaptive equalizers for QAM communication systems based on convex cost functions. *SPIE Optical Engineering*, 31(6):1189–1199, June 1992.

[35] O.L. Frost III. An an algorithm for linearly constrained adaptive array processing. *Proc. IEEE*, 60(8):926–935, August 1972.

[36] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Sydney, 1990.

[37] L.J. Griffiths and C.W. Jim. An alternative approach to linearly constrained adaptive beamforming. *IEEE Trans. on Antennas and Propagation*, 30(1):27–34, January 1982.